We return to the issue of finding a particular solution y_p to the nonhomogeneous second order linear constant coefficient ODE: equation $L[y] = g(t)$, where

$$L[y] = ay'' + by' + cy, \qquad a \neq 0$$

Recall that to find the general solution to this ODE we merely need to add on the complementary solution y_c, which is the general solution to the associated homogeneous ODE which we have now determined regardless what type of root the characteristic polynomial may have.

We shall not try to deal with the nonhomogeneous ODE with the most general g. Instead we deal with 5 cases which are amenable to a uniform approach which has become known as "method of undetermined coefficients". These 5 cases fortunately include some nonhomogeneous ODE's with practical applications and will be discussed in detail after the first exam. For now we simple list the five possibilities:

i. $g(t)$ is a polynomial of degree n, where n is a nonnegative integer.
ii. $g(t)$ is an exponential function $e^{\alpha t}$.
iii. $g(t)$ is either a sine or cosine function, $\cos\beta t$, $\sin\beta t$.
iv. $g(t)$ is a product of **ii.** and **iii.** .
v. $g(t)$ is a product of **i.** and **iv.** .

The "method of undetermined coefficients" is justified mathematically by the following (not so simple) formula:

$$L[t^n e^{rt}] = \left(t^n E(r) + nt^{n-1}E'(r) + \frac{n(n-1)}{2}t^{n-2}E''(r)\right)e^{rt}$$

where $E(r)$ is the characteristic polynomial for $L[y]$. If you wish to know about the derivation and use of this formula then click here.

We will partially cover cases **i.** and **ii.** today, **iii.** and **iv.** tomorrow and at that point they will appear to be unified into one procedure. However, the complete application of this method requires special action in an excpetional situation which will only be covered next week.

This method hinges on our ability to predict the outcome of plugging certain types of function into the left hand side of a linear constant coefficient ODE:

For example, in the first case the right hand side of the ODE $g(t)$ is a polynomial of degree n. If one plugs a generic polynomial of degree n into the left hand side of the ODE, the result obviously will be another polynomial of degree no higher than n. Therefore, if we are good at algebra, then perhaps we can choose the coefficients of the generic polynomial to match the coefficients of the given $g(t)$.

We already saw a successful example of this calculation when we found y_p for the ODE $y'' + y' - 2y = 4$ by plugging in $y = k$(constant). (Recall that a constant is a polynomial of degree zero.)

Let us look at a more sophisticated example to see if we can indeed do the required algebra. So consider

Example 1: $y'' - 3y' - 4y = t^2$

Let us try to find the coefficients of the generic polynomila of degree 2 $At^2 + Bt + C$ that makes y_p into a solution of this nonhomogeneous ODE:

$$\begin{aligned}
-4\,(y_p &= At^2 + Bt + C) \\
-3\,(y_p' &= 2At + B) \\
1\,(y_p'' &= 2A) \\
L[y_p] &= -4At^2 + (-4B - 6A)t - 4C - 3B + 2A
\end{aligned}$$

So we determine y_p by equating coefficients of cor in $L[y_p]$ and $t^2 + 0t + 0$

$$\begin{aligned} -4A &= 1 \\ -4B - 6A &= 0 \\ -4C - 3B - 2A &= 1 \\ L[y_p] &= t^2 + 0t + 0 \end{aligned}$$

We see that $A = -1/4$, $B = -(6(-1/4))/4 = 3/8$, $C = -(1 + 2(-1/4) + 3(3/8))/4 = 13/32$, ie, $y_p = \frac{-1}{4}t^2 + \frac{3}{8}t + \frac{13}{32}$. and $y = c_1e^{4t} + c_2e^{-t} + y_p$ is the general solution of this nonhomogeneous ODE. Generally, three equations in three unknowns are solvable. Unfortunately, there are exceptions but we will postpone looking at these. Other than these exceptions the above example summarizes case **i.** fairly well.

We now go on to an example of **ii.**:

Example 2: $y'' - 3y' - 4y = 5e^{6t}$

Here we see that in order to produce an exponential function e^{6t} by plugging something into the left hand side of the the ODE we need to plug in the same exponential, perhaps multiplied by some constant. Let's see exactly why by plugging into to the left hand side of the above ODE the "guess" $y_p = Ae^{6t}$:

$$\begin{aligned} -4(y_p &= Ae^{6t}) \\ -3(y_p' &= 6Ae^{t}) \\ 1(y_p'' &= 36Ae^{t}) \end{aligned}$$

Therefore, $(-4A - 18A + 36A)e^t = 2e^t$ and hence $14A = 5$ which allows us to conclude that and $y_p = \frac{5}{14}e^{16t}$ The equation for the constant A will generally turn out to be easy to solve. However, there will be exceptional cases is with this sort of example as well.

We now consider the possibility of finding a particular solution y_p in case the function $g(t)$ is a product of a polynomial of degree n and an exponential function e^{rt}. Remember that the constant 5 is a polynomial of degree zero. So the example we just looked with $g(t) = 5e^{6t}$ is an example of the what we wish to consider now. Moreover, the polynomial t^2 is actually equal to t^2e^{0t}. So the right hand side of the first ODE we considered in the first example is also an example, in some sense a trivial example, of the situation where $g(t)$ is a polynomial of degree n time an exponential e^{rt}.

After these remarks, which although trivial help begin to paint a unified of the five cases under consideration:

Example 3: Find a particular solution to $y'' - 3y' - 4y = (4t + 1)e^{3t}$

ANS. We expect a solution of the form $y_p = (At + B)e^{3t}$ Plugging in:

$$\begin{aligned} -4(y_p &= (At + B)e^{3t}) \\ -3(y_p' &= (A + 3(At + B))e^{3t})) \\ 1(y_p'' &= 3A + 3(A + 3(At + B))e^{3t}) \end{aligned}$$

Setting all terms involving the factor t equal gives $-4A - 9A + 9A = 4$ and hence $A = -1$. Now all terms not involving the factor t must add up to $1e^{3t}$. Hence $-4B - 3A - 9B + 3A + 3A + 9B = 1$. This means that $-4B + 3A = 1$, ie, $4B = 3A - 1 = -4$. So $B = -1$. Therefore, $y_p = (-t - 1)e^{3t}$