Exercises 34-42 in Section 3.3 of your textbook are related to the the so-called Euler ODE:

$$at^2 y'' + bty' + cy = 0, \qquad a, b, c \text{ are constants} \qquad t > 0.$$

Since these ODE's are solved in a manner very similar to the linear constant coefficient homogeneous ODE's that we have been working on in this chapter, it is worthwhile having a look at them as well. In order to be able to completely discern the connections between the two, it is necessary to use a different symbol for the independent variable in this discusssion. Let's use x. That is, we consider

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{d^2 y}{dx^2} + cy = 0, \qquad a, b, c \text{ are constants} \qquad t > 0.$$

This linear homogeneous ODE has the coefficients which are constants multiplied by decreasing powers of x as one reads the equation from left to right. One suspects that this equation may have a solution of the form $y = x^r$. This can be verified by plugging in $y = x^r$ and observing that the result is a polynomial times x^r. Polynomial is called the indicial polynomial.

So let us try to find the indicial polynomial. Set $y = x^r$. Then $\frac{dy}{dx} = rx^{r-1}$ and $\frac{d^2 y}{dx^2} = r(r-1)x^{r-2}$. Now plug this into the Euler ODE above:

$$L[x^r] = ax^2 \left(r(r-1)x^{r-2} \right) + bx \left(rx^{r-1} \right) + c \left(x^r \right) = 0$$

Factoring x^r from each term gives the following

$$\left(ar(r-1) + br + c \right) x^r = 0$$

That is the indicial polynomial is

$$ar^2 + (b-a)r + c$$

Consider the following example to see how this works in practice.

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - 8y = 0, \qquad t > 0.$$

Then the indicial polynomial is $r^2 + (-1-1)r - 8 = (r-4)(r+2)$ and we can easily check by plugging into the Euler ODE that $y_1 = x^4$ and $y_2 = x^{-2}$ are two solutions, as predicted. By viewing their graphs we conclued that they are not multiples of each other. So the general solution is that $y = c_1 x^4 + c_2 x^{-2}$.

Now let's look at an example where the indicial polynomial has double roots:

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0, \qquad t > 0.$$

Indeed, rhe indicial polynomial is $r(r-1) - r + 1$ which has a double root $r_1 = 1$. Thus we see that $y_1 = x$ solves this Euler ODE. However, at this point we have no idea about another solution which is not just a constant multiple.

So we appeal to Abel's formula for the Wronskian

$$W = \exp \left(- \int -x^{-1} \, dx \right) = C_1 x$$

and we choose $C_1 = 1$.

Now according to the defintion of the Wronskian of $y_1 = x$ and y_2 is

$$W(t, y_2) = \det \begin{pmatrix} x & y_2 \\ 1 & \frac{dy_2}{dx} \end{pmatrix} = x \frac{dy_2}{dx} - y_2 = x$$

This gives an easily solved first order linear ODE for y_2:

$$\frac{dy_2}{dx} - \frac{1}{x} y_2 = 1$$

The integrating factor for this linear ODE is $\mu = x^{-1}$ Therefore

$$\frac{d}{dx}\left(y_2 x^{-1}\right) = x^{-1}$$

We have

$$y_2 = x \ln x + Dx$$

or simply $y_2 = x \ln x$

The similarity between the constant coefficient linear ODE and Euler's ODE is remarkable and for this reason one suspects a connection between the two. In fact, the connection is explicitly suggested by the fact that in the case of double roots for the characteristic polynomial we multiply the first solution by a t to get the second one and if the indicial polynomial has double roots we multiply the first solution by $\ln x$ to get the second. Indeed let us take the last Euler equation and apply to it the following substitution for the variable x.

$$t = \ln x \qquad \text{or} \qquad \text{equivalently} \qquad x = e^t \qquad x > 0.$$

We use the chain rule to compute the derivative of y with respect to x in terms of the derivative of y with respect to t and the derivative of t with respect to x:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{1}{x}$$

We find the formula relating the the second derivative of y with respect to the two different variables by applying the product rule and chain rule to the above

$$
\begin{aligned}
\frac{dy^2}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{1}{x} + \frac{dy}{dt}\frac{d}{dx}\frac{1}{x} \\
&= \frac{dy^2}{dt^2}\frac{1}{x^2} - \frac{dy}{dt}\frac{1}{x^2}
\end{aligned}
$$

Now let's plug these into the Euler ODE above rewritten here:

$$x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + y = 0, \qquad t > 0.$$

We get

$$x^2\left(\frac{dy^2}{dt^2}\frac{1}{x^2} - \frac{dy}{dt}\frac{1}{x^2}\right) - x\left(\frac{dy}{dt}\frac{1}{x}\right) + y = 0$$

Simplifying gives

$$\frac{dy^2}{dt^2} - 2\frac{dy}{dt} + y = 0$$

which has two solution $w_1 = e^t$ and $w_2 = te^t$ and which confirms the equivalence of the two ODE's when subjected to this substitution of variables.