Having defined the complex exponential function and having seen that if satisfies the same differentiation rule as the real exponential function:

$$\frac{d}{dt}e^{\gamma t} = \gamma e^{\gamma t}$$

enables us to immediately solve those linear homogeneous constant coefficient whose characteristic polynomial has complex roots.

So we consider a linear constant coefficient homogeneous second order ODE which we write as

$$ay'' + by' + cy = 0$$

We assume that the the characteristic polynomial $ar^2 + br + c$ has complex roots γ and $\overline{\gamma}$. Also in this case a fundamental pair y_1, y_2 can be obtained by taking real and imaginary parts of of $e^{\gamma t}$. This is true for the same reason that it was true when γ was real.

Here, since

$$\gamma = \alpha + i\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

we see that $L[e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t] = 0$. And then by the superposition principle

$$0 = L[e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t] = L[e^{\alpha t}\cos\beta t] + iL[e^{\alpha t}\sin\beta t]$$

Since the entire linear combination in the above formula is zero, both the real and imaginary parts individually are zero. This says that if y_1 denotes $e^{\alpha t}\cos\beta t$ and y_2 denotes $e^{\alpha t}\sin\beta t$, then each one has the property $L[y_1] = 0$ and $L[y_2] = 0$. (Although in the real case we could use y_1 and y_2 for either solution, here it is better to associate y_1 with the solution have the cosine function in it.) That is, we have constructed two solutions to the ODE from the one complex-valued solution. Hopefully, they are not multiples of each other. That they are not multiples, can be seen by comparing their values at $t = 0$ where $y_2 = 0$ and $y_1 \neq=$ and at $t = \frac{\pi}{2\beta}$ where $y_1 = 0$ but $y_2 \neq 0$.

We now illustrate the above ideas by solving

$$y'' + 14y' + 74y = 0, \qquad y(0) = 11, \qquad y'(0) = -12$$

The characteristic polynomial is $r^2 + 14r + 74$ and we find its roots by completing the square (much more reliable than using the quadratic formula): $r^2 + 14r + 7^2 = -74 + 7^2$, ie, $(r+7)^2 = -25$. Therefore the roots are $\gamma = -7 \pm i5$. A complex solution is: $e^{(-7+i5)t} = e^{-7t}(\cos(5t) + i\sin(5t))$. Taking y_1 and y_2 to be its real and imaginary parts we obtain

$$y_1 = \operatorname{Re}(e^{-7t}(\cos(5t) + i\sin(5t)) = e^{-7t}\cos 5t \qquad y_2 = \operatorname{Im}(e^{-7t}(\cos(5t) + i\sin(5t)) = e^{-7t}\sin 5t$$

The general solution can be written more neatly as follows:

$$y = e^{-7t}(c_1\cos 5t + c_2\sin 5t)$$

Its derivative is:

$$y' = -7y + 5e^{-7t}(-(c_1\sin 5t + c_2\cos 5t)$$

Plugging in the initial conditions gives

$$y(0) = c_1 = 11, \qquad y'(0) = -77 + 5c_2 = -12 \qquad 5c_2 = 65 \qquad c_2 = 13$$

Therefore, the solution to the IVP is:

$$y = e^{-7t}(11\cos 5t + 13\sin 5t)$$

At this point we should make some observations about the long time behavior of solutions.

The solutions in the complex case involve trig functions which are periodic (in fact, since the variable in the trig function is βt, the period is $2\pi/\beta$) but the presence of the exponential with a negative coefficient for t makes its graph oscillate forever between two decaying exponentials. The only periodic feature of the trig functions that remains in the graph of these solutions is the time between consecutive crossing of the t axis which is always π/β (because there are two crossings of the t axis in each complete cycle) and there are infinitely many of them.

It is important to observe how the solution to this ODE changes as b or equivalently $\alpha = -b/2a$ varies. If b is replaced by its negative, then the oscillations of the solution grow exponentially as t increases without bound. If

β is zero then the exponential function disappears from y_1 and y_2 and hence $y = c_1 y_2 + c_2 y_2$ is periodic with the same period $2\pi/\beta$. We will see later that such a combination of trig functions can be rewritten using a single cosine function the is shifted and multiplied by the amplitude of the oscillations.

Now let us summarize the long time behavior of the solutions of any 2nd order constant coefficient linear homogeneous ODE just by looking at the roots of its characteristic polynomial (except for the repeated roots case which we will deal with next time).

If the characteristic polynomial has complex roots we saw three different types of long time behavior are possible depending on the real part of the complex roots. If the real part is zero we have period solutions (oscillations with constant amplitude), if the real part is positive the solution oscillates forever between two decaying exponential functions and if the real part is negative the solution oscillates forever between two growing exponential functions. In all cases the intervals between crossings of the t-axis are of equal length and hence there are infinitely many of crossings.

If the characteristic polynomial has two positive roots then all solutions decay (approach zero) with increasing t. If the characteristic polynomial has two positive negative roots then eventually all nonzero solutions increase or decrease without bound as t increases without bound, depending on the initial data (which determines c_1 and c_2). Finally, if the characteristic polynomial has one positive and one negative root, then in addition to the above behavior a solution can also decay as t increases without bound.

The case of double roots will be dealt with tomorrow.