Math 251 Sections 1,2,10 Lect Notes 9/18/09

Now that all the theoretical machinery for dealing with solving all constant coefficient 2nd order linear homogeneous ODEs is in place, we return to the actual solution techniques. So far we have only covered the case where the characteristic polynomial has two real roots. To proceed to the case where it has complex roots we need to construct a complex valued exponential function.

Unfortunately the name complex number brings the wrong mental image to people's minds. Many people identify the word "complex numbers " with "complicated", "convoluted", "confusing" which put up uneeded mental road-blocks. Perhaps the following words are more appropriate to associate with "complex numbers": "clear", "clean", "concise" which more accurately reflects the role of complex numbers in applied mathematics.

Recall that a complex number γ is an object formed by a pair of real numbers α and β which are joined by a plus sign and the imaginary unit i in between. (The word "imaginary" unit also has an unfortunate connotation. However, the reality is that i is not less relevant to applied mathematics than $\sqrt{2}$ is. In fact, they are both constructed in mathematics by the same procedure and are both essential for enabling mathematics to solve real world problems.)

Complex numbers are added, subtracted, multiplied and divided by invoking the usual rules of algebra, together with the additional rule: $i^2 = -1$. Thus $(5+6i)+(3+4i) = 8+1i0$, $(5+6i)(3+4i) = (15-24)+i38 = -9+i38$, $1/(3+4i) = (3-4i)/25$. The latter calculation is used when dividing one complex number by another because division is usually replaced by multiplication by the reciprocal.

There are several additional items of notation: for any complex number $\gamma = \alpha + i\beta$ the complex conjugate is $\overline{\gamma} = \alpha - i\beta$. The absolute value of a complex number γ is the distance of the point (α, β) to the origin. For this reason it is also called the magnitude of γ. And its value can be written in two different ways $\sqrt{\overline{\gamma}\gamma}$ or $\sqrt{\alpha^2+\beta^2}$.

We say that real part of γ is α, in symbols, $\text{Re}\gamma = \alpha$ and the imaginary of γ is β, in symbols, $\text{Im}\gamma = \beta$. Please note that according to the definition the Imaginary part of a number is real. If one feels that somehow this is not the right way to make this definition then that feeling must be set aside because writing the answer $\text{Im}(3+i4) = i4$ on a quiz cannot receive credit, whereas $\text{Im}(3+i4) = 4$ can!

Connected with the notion of absolute value we also have the argument of a complex number $\arg(\gamma)$ which is the angle between the line joining the point (α, β) to the origin and the positive horizontal axis in the plane, also called the positive real axis. An interesting geometric interpretation of multiplication of complex numbers is that when two complex numbers are multiplied, then their absolute values are multiplied and their arguments are added (modulo 2π).

We now proceed to defining the exponential of a complex number. Your textbook gives an elaborate derivation based on the power series of sine, cosine and the exponential. However, since the concept of convergence is totally ignored in this derivation, it essentially is without meaning. It is much more honest to say we are making a definition, as frequently is done in mathematics, and this definition when combined with the definition of derivative of a complex-valued function of a real variable leads us to very simple solution to the ODE we are now seeking to solve.

This definition comes in two parts: for any real number β

$$e^{i\beta} = \cos\beta + i\sin\beta$$

And for any complex number $\gamma = \alpha + i\beta$

$$e^{\gamma} = e^{\alpha+i\beta} = e^{\alpha}e^{i\beta}$$

Our consideration will be focused on the following complex exponential function:

$$e^{\gamma t} = e^{(\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t} \left(\cos \beta t + i \sin \beta t\right)$$

This is a complex-valued function of the variable t; ie, for each real number t the value assigned to t is a complex number. We need to know how to differentiate such a function and the answer is given by the following definition:

$$\text{If} \quad f(t) = g(t) + ih(t) \quad \text{then} \quad f'(t) = g'(t) + ih'(t)$$

We apply this to the function $e^{\gamma t}$:

$$\frac{d}{dt} e^{\gamma t} = \frac{d}{dt} e^{\alpha t} \cos \beta t + i \frac{d}{dt} e^{\alpha t} \sin \beta t = \left(\alpha e^{\alpha t} \cos \beta t - \beta e^{\alpha t} \sin \beta t\right) + i \left(\alpha e^{\alpha t} \sin \beta t + \beta e^{\alpha t} \cos \beta t\right)$$

For this calculation to be useful to us we must have the usual rule for differentiation

$$\frac{d}{dt} e^{\gamma t} = \gamma e^{\gamma t}$$

Since we just finished expanding the lhs of this equation, to verify its validity we simply need to carry out the following multiplication:

$$\left(\alpha + i\beta\right) e^{\alpha t} \left(\cos \beta t + i \sin \beta t\right)$$