We continue with an important observation related to solving nonhomogeous ODEs and an amazing formula involving the Wronskian.

Up til now the 2nd linear constant coefficient ODE's we solved were limited to homogeneous ODE's. However, we will see that there are some significant applications of these ODEs that requires the solution to non homogeneous ODEs as well. So here we will give a glimpse of how this will be accomplished, but we will not tell the entire story until we complete our dealings with the homogeneous ones.

Consider the nonhomogeneous linear 2nd order ODE $y''+py'+qy=g$. The ODE which is identical with this except that the g is replace by zero is called the complementary, or associated, ODE. Note that both of these can be written with our $L[y]$ notation. The nonhomogeneous one is $L[y]=g$ and the associated homogeneous ODE is $L[y]=0$. Now let us adopt the notation y_c for the general solution $c_1y_1c_2y_2$ of the ODE $L[y]=0$. Frequently y_c is called the **complementary solution**. (Note that the words "complementary" or "associated" are not used by everyone; thus you need to be aware of the context when reading new material on this topic.) Suppose that somehow, perhaps even by making a lucky guess, we find y_p, a **particular** solution to $L[y]=g$. Then we see by the Superposition Principle that y_p+y_c is a solution of the nonhomogeneous ODE, ie, $L[y_p+y_c]=L[y_p]+L[y_c]=g+0=g$. Also suppose we are given an IVP for the nonhomogeneous ODE: $y(t_0)=\alpha$ and $y'(t_0)=\beta$. Then plugging in t_0,α and β leads us to the following linear system of algebraic equations:

$$\begin{aligned} y_p(t_0)+c_1y_1(t_0)+c_2y_2(t_0) &= \alpha \\ y_p'(t_0)+c_1y_1'(t_0)+c_2y_2'(t_0) &= \beta \end{aligned}$$

We can be conveniently rewritten as

$$\begin{aligned} ccc_1y_1(t_0)+c_2y_2(t_0) &= \alpha_1 \\ c_1y_1'(t_0)+c_2y_2'(t_0) &= \beta_1 \end{aligned}$$

$$\text{where} \qquad \alpha_1=\alpha-y_p(t_0) \qquad \text{and} \qquad \beta_1 = \beta-y_p'(t_0)$$

But last time we observed that for a fundamental set y_1, y_2, the above system of linear algebraic equations can be solved for any choice of α,β, including the choice α_1,β_1.

So let us use this observation to solve the IVP $y(0)=1$ and $y'(0)=2$ for the nonhomogeneous ODE $y''+y'-2y=4$. Recall that $y_c=c_1e^t+c_2e^{-2t}$ is the solution to the associated homogeneous ODE. Now let's guess at y_p. Perhaps we could even guess that a constant would be a solution. Then it is fairly clear that the constant would have to be -2; that is we have guessed that $y_p=-2$ is a particular solution. So we need to plug in $t=0$, $y=1$ and $y'=2$ into $y=c_1e^t+c_2e^{-2t}-2$ and $y'=c_1e^t-2c_2e^{-2t}$. That is, $1=c_1+c_2-2$ and $2=c_1-2c_2$. Subtracting the second from the first gives $-1=3c_2-2$. That is, $c_2=1/3$ and $c_1=8/3$. So finally, the solution is $y=\frac{8}{3}e^t+\frac{1}{3}e^{-2t}-2$.

The above result tells us the key to solving IVPs for the nonhomogeneous linear ODE hinges on our ability to do it for the homogeneous one. This leads us to question whether a given fundamental set will work for one specific value of t_0 but not for another choice of t_0. We saw that in the constant coefficient case we could freely shift shift solutions back and forth, which places t_0 into a very minor position of influence. However, what about for nonconstant coefficients? Can it happen that a pair of solutions manages to be a fundamental set for one choice of t_0 but not for another?

The answer is given by an amazing formula known as Abel's formula which we now formulate. Suppose that y_1 and y_2 are two solutions of the homogeneous linear ODE $y''+py'+qy=0$ and that p and q are continuous on an open interval I. This means that

$$y_1''+py_1'+qy_1 = 0 \qquad (1)$$

$$y_2''+py_2'+qy_2 = 0 \qquad (2)$$

We would like to eliminate the coefficient q from these two equations if possible. It is if we multiply equation (1) by y_2,and subtract from it equation (2) multiplied by y_1. If we do this we obtain

$$y_2y_1'' - y_1y_2'' + p(y_2y_1' - y_1y_2') \quad = \quad 0 \tag{3}$$

We recall that $W(y_1, y_2) = y_1y_2' - y_1'y_2$.
We differentiate this by applying the product rule to each of the two terms on the right:

$$\frac{d}{dt}W(y_1, y_2) = y_1'y_2' + y_1y_2" - (y_1"y_2 + y_1'y_2') = y_1y_2'' - y_1"y_2$$

We now plug this information into preceding formula (3) and rewrite it as:

$$0 = y_2y_1'' - y_1y_2'' + p(y_2y_1' - y_1y_2') = \frac{d}{dt}W(y_1, y_2) + pW(y_1, y_2)$$

This says that the Wronskian of two solutions itself is a solution of a very simple first order ODE: $W = pW'$. This is a very simple separable ODE with solution given by:

$$W = Ce^{-\int p\,dt} \tag{4}$$

(NOTE: the only difficulty with this formula is its close resemblence to the formula for the integrating factor in the first order linear situation. Because of this, extreme care must be taken not to confuse them.) Then Abel's formula is

$$W(y_1, y_2)(t) = Ce^{-\int p\,dt}$$

Among the possible choices for C is the following:

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)e^{-\int p\,dt} \tag{5}$$

This, among many things, tells us what we wanted to know: if $W(y_1, y_2)(t_0) \neq 0$, then $W(y_1, y_2)(t) \neq 0$ for any t in the open interval I. ie, if the pairs y_1, y_2 serves as a fundamental pair for t_0m then it will work to for any t_0 in the entire interval I.

Abel's formula also tells us that the Wronskian is really determined by p the coefficient of y' and not by q, It also gives us a procedure for finding a second solution y_2 once we know one solution to a linear homogeneous ODE. (Of course, if our objective is to produce a fundamental pair of solutions it serves no useful purpose to take the second solution to be a constant multiple of the first.) To illustrate this procedure we consider the so-called Euler equation: $t^2y'' + 4ty' - 4y = 0, \quad t > 0$ For this linear ODE it is rather easy to guess one solution: $y_1 = t$. So we rewrite this Euler ODE in the form to which Abel's formula applies (coefficient of y" is one): $y'' + \frac{4}{t}y' - \frac{4}{t^2}y = 0$. By Abel's formula we see that

$$W(y_1, y_2) = Ce^{-\int 4/t\,dt} = 1/t^4 \tag{6}$$

where we make the simplifying assumption that $W(y_1, y_2)(1) = 1$. (Once we see our final answer for y_2 it is worthwhile to repeat this calculutaion without the simplifying assumption to be sure that the final answer is not changed.) Plugging in $y_1 = t$ into the definition of the Wronskian gives

$$det\begin{pmatrix} t & y_2 \\ 1 & y_2' \end{pmatrix} = ty_2' - y_2$$

Equating this with the result we obtained in formula (6) gives the new 1st order ODE for y_2:

$$ty' - y = t^{-4} \qquad \text{or} \qquad y' - \frac{1}{t}y = t^{-5}$$

The integrating factor for this ODE is $\mu = t^{-1}$ and hence

$$\left(yt^{-1}\right)' = t^{-6} \qquad \text{So} \qquad yt^{-1} = \frac{1}{-5}t^{-5} + C_2 \qquad \text{So} \qquad y = \frac{1}{-5}t^{-4} + Ct$$

Although y is an acceptable second solution, it is worthwhile to take a moment to simplify it. First we multiply it by -5 and then we add to it $5Cy_1$ to finally obtain $y_2 = t^{-4}$.

Now that we are about to embark on a major effort at solving second order linear ODE's, it is worthwhile to state the instructions needed to get Maxima to solve them under xMaxima:

(%i1) eqn: ‘diff$(y, t, 2) + 2*$’diff$(y, t) + 2 * y$;
(%i2) sln: ode2(eqn, y, t);
(%i3) ic2(sln, $t = 0, y = 1$,diff$(y, t) = 1)$;

Note that one of the packages of procedures for solving 1st and 2nd order ODE's available to Maxima is callled ode2 and that it is loaded by default. The instruction ic2 is reserved for solving IVP once the general solution containing two unknown constants, e.g., %k1 and %k2, has been found. Note that in the above dialog with Maxima, everytime a derivative appears it is preceded by an apostrophe, namely ‘diff$(y, t) = 1$. This indicates to Maxima that it is an unknown and a value for it cannot be found until Maxima completes the process of solving the ODE.