Math 251 Sections 1,2,10 Lect Notes 9/16/09

Today we cover several theoretical aspects that are helpful in applications. The first is the existence and uniqueness guarantee for secont order linear ODE's.

For this we assume that the ODE is written as $1y'' + py' + qy = g$, where p, q and g are functions of t and the IVP for this ODE is $y(t_0) = \alpha$ and $y'(t_0) = \beta$. If I is an open interval containing t_0 but not any discontinuities of p, q and g, then a unique solution exists on the entire interval I.

Let us see how this could be applied. For this purpose let us try to determine the largest interval on which the following initial value problem has a unique solution:
$(t^2 - 4)y'' + (t-2)y' + 3(t+2)y = \frac{1}{t-1}, \quad y(0) = 1, \quad y'(0) = 2$

To answer the question we rewrite the ODE so that the coefficient of y'' is 1: $y'' + \dfrac{1}{t+2)}y' + \dfrac{3}{(t-2)}y = \dfrac{1}{(t-2(t-1)(t+2))}$. The coefficients p, q, and g are discontinuous at $-2, 1, 2$. The largest open interval contanig $t = 0$ not containing the above discontinuities is $(-2, 1)$.

This is such an obvious adaptation of the first guarantee given for the first order case, that you may be wondering why do we bother to state it instead of just saying that it is an "obvious adaptation". Actually, the content is subtly but significantly different in the two cases. And this is what should be emphasized: in the first order case two solutions could not have graphs that intersect at a point (t_1, y_0) as long as the two functions p and g are continuous at t_1 (Recall, that at a point of continuity the solution to an IVP problem is unique. If two solutions intersect at (t_1, y_1) then applying the uniqueness guarantee with this point as the initial condition implies that the two solutions are one. This contradiction shows that graphs in fact cannot intersect at points of continuity of p and g.

However, in the second order case we saw that the ODE the second order $y'' + y' - 2y = 0$ has two solutions $y_1 = e^t$ and $y_2 = e^{-2t}$ which obviously intersect at the point $(0, 1)$. This apparent anamoly disappears however when one realizes that initial conditions for 2nd order ODE's involve two conditions: $y(t_0) = \alpha$ and $y'(t_0) = \beta$ Therefore, in the second order case we conclude that two solutions cannot have graphs that intersect at a point (t_0, α) and have the same slope β there, as long as the two functions p and g are continuous near t_0

You will probably be relieved to hear that we will not present any existence and uniqueness guarantees for the 2nd order nonlinear ODE.

However, as another example of the meaning of the linear 2nd order guarantee we answer why $y_1(t) = t^2$ is not a solution to any homoegeneous linear second order ODE with coefficients continuous near 0. The key to explaining this behavior (or lack of it) is the fact that the zero funtion $y_2 = 0$ is must be a solution to a homogeneous ODE. Furthermore y_1 and y_2 have the same values at $t = 0$ and the same holds true for their derivatives. Therefore the uniqueness guarantee says that only one of them can solve the ODE. But as we said there is no choice about the zero funtion y_2. That is whay y_1 is not a solution.

We now consider another issue: for which pairs of solutions y_1, y_2 of a homogeneous 2nd order linear ODE's do we have the ability to solve any IVP. The answer is obviously not "for every pair". Conside the situation where $y_2 = 17y_1(t)$ then $c_1y_1 + c_2y_2 = (c_1 + 17c_2)y_1 = Cy_1$. So if $y_1(t_0)$ happens to be 0 then we cannot solve for examply the IVP with $\alpha = 1$. On the other hand if $y_1(t_0)$ happens to nonzero, then we cannot solve the the IVP with $\alpha = 0, \beta = 1$.

To answer the question posed in the previous paragraph we assume that y_1 and y_2 are two solutions and try to find c_1 and c_2 so that the linear combination $y = c_1 y_1 + c_2 y_2$ satisfies the given IVP $y(t_0) = \alpha, y'(t_0) = \beta$: i.e.,

$$\begin{array}{rcl} y(t_0) = & c_1 y_1(t_0) + c_2 y_2(t_0) = & \alpha \\ y'(t_0) = & c_1 y_1'(t_0) + c_2 y_2'(t_0) = & \beta \end{array}$$

In order to find c_2 we eliminate c_2 by subtracting $y_2(t_0)$ times the 2nd equation from $y_2'(t_0)$ times the first equation:

$$c_1 \left(y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \right) = \text{some} \quad \text{number}$$

The only conceivable obstacle to determining the value of c_1 is the possiblility of the stuff inside the parenthesis being zero. For this reason we assign a special name to that stuff: Wronskian of y_1, y_2 evaluated at t_0. And we also assign to it the special symbol $W(y_1, y_2)(t_0)$. Therefore whenever $W(y_1, y_2)(t_0) \neq 0$ we are assured that for every α and β the constants c_1 and c_2 which give a solution to the IVP can be found.

A pair of solutions y_1, y_2 with this property is called a **fundamental pair** or **fundamental set**. The linear combination $y = c_1 y_1 + c_2 y_2$, which solves any conceivable IVP for this ODE is called the **general solutions**.

This explains how we could be so confident in dealing with the IVP immediately after we discovered last time that $y_1(t) = e^t$ and $y_2(t) = e^{-2t}$ are solutions of $y'' + y' - 2y = 0$. Indeed if we compute their Wronskian we see that it is never zero.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & e^{-2t} \\ e^t & -2e^{-2t} \end{vmatrix} = -3e{-}t$$

Here the absolute value signs enclosing the 2×2 array denote the determinant of the array. So indeed the Wronskian does not vanish and the IVP can be solved for any value of t_0, α and β.

Frequently, one can spot functions which are multiples of each other and for these one need not go to the trouble of finding their Wronskian in order to conclude that they do not form a fundamental set. In particular, the function 0, although a universal solution to linear homogeneous ODEs, can never be part of a fundamental set.

For example it is easy to pick a pair of functions from the following collection of functions that might be a fundamental set of solutions for some homogeneous differential equation.

$$0, \quad e^{3t}, \quad e^{3t+4}, \quad e^{t/3}$$

Since e^{3t+4} is a multiple of e^{3t} only one of them could sever in a fundamental set formed by pairs of functions from the above list and 0 can never be part of a fundamental set. This leaves two possibilities.

This leads us to suspect that there maybe some prefernce for fundamenntal sets that have some neat properties. One possible preference is for a pair of solutions w_1 and w_2 that have the properties $w_1(0) = 1, y_1'(0) = 0, y_1(0) = 0, y_2'(0) = 1$ In the case $y_1 = e^t$ and $y_2 = e^{-2t}$ the pair that meets this preference would be

$$w_1(t) = \frac{2}{3}e^t + \frac{1}{3}e^{-2t} \qquad w_2(t) = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$$

(Note: we actually found $w_1(t)$ during the last class and the process for finding w_2 is entirely analogous.)

Finally the reason one would have for this preference is that an arbitrary IVP $y(0) = \alpha$ and $y'(0) = \beta$ can now be solved immediately by writing $\alpha w_1 + \beta w_2$ eliminating the need to solve a new system of algebraic each time a new IVP comes along.