We now leave 1st order ODE's (for a little while) and turn our attention to 2nd order ODE's. The general 2nd order ODE in this course will be assumed to be expressible as:

$$y'' = f(t, y, y')$$

In the 1st order case we anticipated at some point that a constant of integration would appear in our solution method and therefore we also sought to solve initial value problem $y(t_0) = y_0$. After we covered the existence and uniqueness guarantees, we saw that this expectation was reasonable. In the 2nd order case we do the same sort of wishful thinking and try to find a solution y that also satisfies the IVP: $y(t_0) = \alpha$, $y'(t_0) = \beta$, for given constants α, β

We will deal primarily with **linear** 2nd order ODE's, i.e. $y'' = -py' - qy + g$ where p, q and g are given functions of t, and any one, two, or even three of them maybe zero. We will say the ODE is **autonomous** if p, q and g are constant functions. On the other hand we will say the 2nd order ODE has **constant coefficients** if p and q are constant but not necessarily g. This hints at the fact that usually we will write our 2nd order linear ODE in the following standard form

$$y'' + py' + qy = g$$

Another bit of terminology is that the 2nd order linear ODE is **homogeneous** if g is the zero function. An equivalent way of describing linear homogeneous ODE's is: a linear ODE is homogeneous if and only if the constant function $y = 0$ is a solution. One needs to be aware of the fact that the word "homogeneous" is a widely used term in mathematics and frequently has has little or no connection to the term that we defined here. For example, on page 49 in the textbook the word "homogeneous" refers to a different idea.

Chapter 3 begin with 2nd order linear constant coefficient homogeneous ODEs. i.e.,

$$ay'' + by' + cy = 0$$

where a is a nonzero constant and b, c are just any constants. This conflicts a bit with the form above $y''+py'+qy = 0$ which required that 1 be the coefficient of y''. The excuse for this inconsistency is that a is required to be nonzero so if need be the transition from one form to other is immediate.

Finally one more bit of notation is that for any given function $w = w(t)$, we denote the result of plugging in w into the lhs (left hand side) of the equation by $L[y]$. In other words we are seeking to solve an equation that can be written very briefly as:

$$L[y] = 0 \qquad \textbf{where} \qquad L[y] = ay'' + by' + cy$$

We this point in the course where our strategy for solving an ODE will be following a hunch, as opposed to a systematic procedure. As we see that our hunch is working for us we will present the mathematical results that justify our conclusions.

Based on our experience in Calculus it is seems that the only function that has a chance of solving the ODE we dealing with is an exponential function because it is the only one that has a derivative that apart from a constant very closely resemble the original function.

Therefore we are inclined try to see if e^{rt} is a solution. That is perhaps we can choose r so that $L[e^{rt}] = 0$. So following this hunch we plug into the ODE:

$$L[e^{rt}] = a\left(e^{rt}\right)'' + b\left(e^{rt}\right)' + c\left(e^{rt}\right) = ar^2e^{rt} + bre^{rt} + ce^{rt} = \left(ar^2 + br + c\right)e^{rt}$$

The result of our little experiment is very interesting. Note that produced a quadratic polynomial ar^2+br+c times an exponential e^{rt}; something that is very supportive of our hunch because this immediately tells us that if we choose the r correctly we can in fact make e^{rt} be a solution. In many cases we can even produce two solutions. This stems from our ability to find all posible roots of all quadratic polynomials. Specifically, sometimes it is possible to factor a quadratic polynomial by inspection and thereby find its roots; but it is always possible to complete the square and find either two real roots, or one repeated root, or two complex roots.

The polynomial $ar^2 + br + c$ is called the **characteristic polynomial**. Please remember this name. You will meet it again in this course and frequently elsewhere in mathematics.

For today let's assume the **characteristic polynomial** has two real roots r_1, r_2. Then we have two solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$.

We now turn to IVPs for this type of equation. Consider a specific example: $y'' + y' - 2y = 0$ and $y(0) = 1$, $y'(0) = 0$. The characteristic polynomial is $r^2 + r - 2 = (r-1)(r+2)$. There $y_1 = e^t$ and $y_2 = e^{-2t}$ Although both y_1 and y_2 satisfies the first half of the IVP, neither one satisfies the second. It is obvious we need to produce more solutions and we recall that arbitrary constant that entered the calculation at the point where we integrated gave us this ability in the first order case. In our current situation we followed a hunch instead of following a systematic procedure; we did not do any integration and we are at loss on where one or two arbitrary constants could be placed in the solution. Eg, if we try to add a constant, then we see that $y_1 + C$ is not a solution of the current ODE. Moreover it does not advance the cause of solving the IVP either.

A very simple idea bearing the name **Superposition Principle** from Physics comes to the rescue. It says that for linear ODE's and any two functions $u = u(t)$ and $w = w(t)$ and any two constants c_1 and c_2 the following holds:

$$L[c_1 u + c_2 w] = c_1 L[u] + c_2 L[w]$$

Although we are now focused on constant coefficient homogeneous 2nd order linear ODE's, the Superposition Principle is valid for any linear ODE (and linear PDE's as well). In fact it makes very frequent appearances in all of mathematics and all of its applications in various different disguises. You in fact encountered for the Superpostion Principle for the first time when you learned the rules of basic arithmetic: The calculation of (2)(3)+(2)(4) can be performed in two different ways: one can first add 3 and 4 and then multiply the result by 2 or one can first multiply each of 3 and 4 by 2 and then add the result. In arithmetic this is called the distributive law. If says if two numbers are to be added and then the result is to multiplied by 2 the answer would be the same as first doing the multiplications by 2 and then adding the results.

Our version of the Superposition Principle says that if we are to take a linear combination of two functions (like $c_1 u + c_2 w$) and then plug the result into to the lhs of a linear ODE, then the answer would be the same as first plugging into the lhs of the linear ODE and then taking the linear combination of the results. The actual verification of the Superposition Principle is very simple and actually it is based on the distributive law and the analogous property for derivatives.

A very simple consequence is the following fact: If y_1 and y_2 are two solutions to a linear homogeneous ODE, then so is any linear combination $c_1 y_1 + c_2 y_2$. Not it is very easy to see exactly why this is true:

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2] = 0$$

Going back to solving the IVP for the homogeneous constant coefficient linear ODE we considered above, we see that a linear combination of the two solutions that we found is also a solution and that linear combination contains c_1, c_2 that we can choose any way we like. Let's see if indeed the linear combination $y = c_1 e^t + +c_2 e^{-2t}$ can be made to solve $y(0) = 1, y'(0) = 0$.

For this purpose we compute $y' = c_1 e^t + -2c_2 e^{-2t}$ and plug in $y = 1, t = 0$ in the formula for y and $y' = 0, t = 0$ in the formula for y'. We have two linear equations with two unknowns c_1, c_2:

$$1 = c_1 + c_2 \qquad\qquad 0 = c_1 - 2c_2$$

Multiplying the first by 2 and adding to the second gives: $c_1 = 2/3, c_2 = 1/3$. That is, the solution to the IVP is

$$\frac{2}{3}e^t + +\frac{1}{3}e^{-2t}$$

One final point. The equation for which we solve the IVP is autonomous. Also in the 2nd order case, this implies that any solution $y(t)$ has the property that $y(t-\gamma)$ is a solution for any gamma. This means that if we are faced with a more complicated IVP like $y(999999) = 1, y'(999999) = 0$, then we simply shift the variable t in our current solution to obtain the solution to the new IVP:

$$\frac{2}{3}e^{t-999999} + +\frac{1}{3}e^{-2(t-999999)}$$

(Note that most calculators will refuse to evaluate exp(999999).)