We have already seen that for separable and linear first order ODEs there are solution procedures which lead to formulas for solutions, provided the integrals met along the way are expressible in terms of formulas. In this course we cover one additional technique which is applicable to all exact ODE's a rather broad collection of ODE's which we will now describe.
With exact ODE's we begin by supposing that the solution has a specific appearance and then try to see if a given ODE fits. More precisely an ODE is called **exact** if its solution has the form

$$F(t, y) = C = \text{const}$$

Here $F(t, y)$ is some function (frequently rather complicated) of the dependent variable y and the independent variable t. Then $F(t, y) = C$ is an equation that represents the general solution of the ODE. We have already seen an example of such solutions when we encountered a separable ODE whose general solution is the family of circles $t^2 + y^2 = C$. In this instance we succeeded in solving for y in terms of t when specific initial conditions were added. But for general exact equation we do not have the hope of being able to express y in terms of t and we will be perfectly happy when we can turn in our work with the final answer as

$$F(t, y) = C \qquad \text{or} \qquad F(t, y) = F(t_0, y_0)$$

where the second formula is the solution satisfying a given initial value condition $y(t_0) = y_0$.
So let us illustrate the idea with the ODE $y' = -t/y$. We would like to put the ODE into the form $F_t + F_y y' = 0$ and then figure out what the F is. With a little algebra we easily rewrite the given ODE as

$$t + yy' = 0$$

Now we seek the function $F = F(t, y)$. We have two requirements of F:

$$F_t = t \qquad \text{and} \qquad F_y = y$$

From the first requiremnt we see that F could be $\frac{1}{2}t^2$. However, this is not the only possibility. In fact, any function of y $h(y)$ could be added to $\frac{1}{2}t^2$ and the requirement would still be met. This is actually very convenient because we still need to verify that $F = \frac{1}{2}t^2 + h(y)$ meets the second requiremnt. Indeed this will be the case if we choose $h(y)$ so that $\frac{dh}{dy} = y$. This means that we should choose $h(y) = \frac{1}{2}y^2$. Putting all this together gives $F(t, y) = \frac{1}{2}t^2 + \frac{1}{2}y^2$. We note that it is not necessary here to write down the constant of integration because the solution we write down must have a constant on the right hand side as well and all constants can be incorporated there:

$$\frac{1}{2}t^2 + \frac{1}{2}y^2 = C \qquad \text{and} \qquad t^2 + y^2 = C_1$$

This was a very simple example. We should really try a more complicated one. But before embarking on that we recall that not every ODE is exact and that causes us some nervousness because we would not like to spend alot of time using this technique to solve an ODE only to discover after making an enormmous effort that it really is not exact. To avoid this waste of effort we need to be aware of one more fact. That is, $(F_t)_y = (F_y)_t$ for functions F that a reasonably behaved.
So when we try to apply the exact ODE solution technique to a new ODE, after identifying candidates for F_t and F_y, but before beginning our search for F, we check whether or not the property $(F_t)_y = (F_y)_t$ holds. And by revesing the implication in the above statement: if F exists, then $(F_t)_y = (F_y)_t$, we obtain a test for exactness. However this test really valid only when the the domain of F is "simply connected". Describing exactly what this means takes us a bit far afield and those of you who are interested can look for more information on this on the internet. In this course this requirement will be met in all our examples.
Now a more complicated example is: $y'(\sin t + t^2 e^y - y^2) = -y\cos t - 2te^y$. Before we can do anything meaningful with this we need to put it into the correct form:

$$y\cos t + 2te^y + (\sin t + t^2 e^y - y^2)y' = 0$$

We would like to find F with the property that $F_t = y\cos t + 2te^y$ and that $F_y = \sin t + t^2 e^y - y^2$. So before we actually seek F we apply the test for exactness: $(F_t)_y = (F_y)_t$, We find that $(F_t)_y = \cos t + 2te^y$ and $(F_y)_t = \cos t + 2te^y$ and the ODE passes the test.

So we look at $F_t = y\cos t + 2te^y$ and figure out that $F = y\sin t + t^2e^y + h(y)$. The next step is to choose $h(y)$ in order to meet the second requiremnt. We get $F_y = \sin t + t^2e^y + \frac{dh}{dy}$ and the second requirement is met by choosing h so $\frac{dh}{dy} = -y^2$. In other words $h(y)$ must be $-\frac{1}{3}y^3$. So here is the solution:

$$y\sin t + t^2e^y - \frac{1}{3}y^3 = C$$

In order to meet an initial condition, say $y(1) = 0$ we set $t = 1$ and $y = 0$ and obtain

$$y\sin t + t^2e^y - \frac{1}{3}y^3 = 1$$