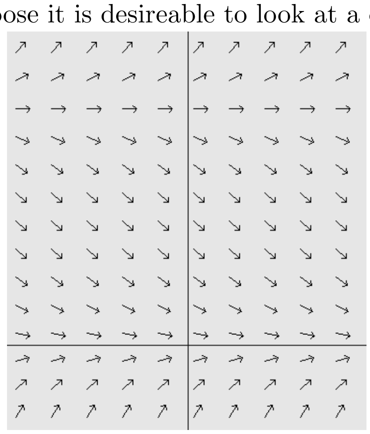
Today we return to the topic of "long time behavior" and use a population problem to motivate the terminology that is introduced.
The ODE $y' = ry$ frequently used to model natural growth. It says that the rate of change of $y$ is proportional to $y$. The constant of proportionality $r$ is the called the **intrinsic growth rate**. This is a little bit of a misnomer, but we have to adjust to it.
We saw at the very beginning of the course that the solution to the above ODE is an exponential function. For this reason the law of natural growth is not an adequate to model for long time behavior of poputalions, because no population can continue to grow exponentially for a long time without an upper bounds. Even a very simple population such as bacteria cannot continue to grow exponentially because before a very long time elapses the entire globe and the entire atmosphere around it would be completely filled and hence the bacteria would run out of nourishment needed to maintain the same growth rate.
An additional ingredient in the ODE is necessary to model the fact that the growth eventually must be limited and even stopped by physical constraints. This leads to the so called logistic model for population growth:

$$y' = ry(1 - (y/a))$$

where $a$ is the limiting size, the size which growth is no longer possible.
For the sake of simplicity let us assume that the variable $y$ represents bacteria in units of $10^6$, the initial rate of growth is 1 and the limiting size is also 1:

$$y' = y(1 - y)$$

Recall that an equilibrium solution is a constant solution $y = c$ and that a critical value (point) of an autonomous ODE, i.e, a value $d$ for which the right hand side of the ODE is zero yields an equilibrium solution $y = d$. The equation above has two critical values 0 and 1 (or, equivalently two equilibrium solutions: $y = 0$ and $y = 1$), Although they are both described as critical right now, they have some entirely different properties, which we will describe, and then attach additional adjectives to remind us of their distinctive properties.
For this purpose it is desireable to look at a direction field for $y' = y(1 - y)$:



To describe the different properties consider the incident of a lab technician who is instructed to clean very thoroughly a petri dish (a dish for growing bacteria) whose capacity is $10^6$ bacteria before leaving the lab one evening. Being in a hurry to leave, the lab technician does not clean the dish thoroughly and just a very very few bacteria remain. The lab technician reports for work the next morning and is immediately fired for leaving a dish full of bacteria and neglecting to carry out assignments.
Now consider the incident of a lab technician who is instructed to very carefully fill a petri dish (a dish for growing bacteria) whose capacity is $10^6$ bacteria with exactly $10^6$ bacteria before leaving the lab one evening. Counting exactly $10^6$ bacteria is somewhat boring so the lab technician the lab technician takes a few short cuts and just approximates a $10^6$ and leaves. The the lab technician reports for work the next morning and is given the employee of the year award for following very difficult orders down to the very precisely.
These two incidents are reflected in the direction field above. For the critical value 0: There are solutions starting "close" to the critical value 0 that eventually move "far" from it.
Whereas for the critical value 1: Every solution that starts sufficiently "close" to the critical value 0 eventually moves "far" from it.
This leads us to the following definitions for critical values $y_0$ of a general autonomous first order ODE: $y = f(y)$.
A critical value $y_0$ is called **unstable** if there is a solution starting "close" to the critical value $y_0$ that eventually

move “far” from it. And a critical value $y_0$ is called **aymptotically stable** if every solution that starts “sufficiently close” to the critical value $y_0$ eventually “approaches” it.
Some remarks are in needed about these definitions are in order. The adjectives unstable and asymptotically stable are not opposites of each other. There is a third property which is the opposite of unstable which we do not consider right now because it does not arise naturally in the present context. It will come up later when we discuss systems of ODEs. In the meantime please be aware that the term **stable** describes a weaker property than the term **asymptotically stable** and for this reason should not be used here. Another remark is that the words “close”, “sufficiently close”, “far”, and “approaches” are left undefined mathematically at the moment. However, your intuitive notion of what they mean should suffice for the working with the examples you will encounter in this course.
Another important concept that the textbook introduces here is formula (8) on page 81. It is a formula for the second derivative of a solution for the ODE:$y' = f(y)$ and hence allows us to visualize the concavity of the solutions that we may wish to graph. For example if we try to graph a solution starting at $(0, .01$, close the unstable critical value 0, then eventually it moves up from it. However at a certain point its concavity must change because it cannot intersect the aysmytotically stable solution $y = 1$. We would like to know where this happens. That is we would like to find inflection points for the solutions.
From the chain rule we see that

$$y'' = \frac{d}{dt}y' = \frac{d}{dt}f(y) = \frac{df}{dy}\frac{dy}{dt} = \frac{df}{dy}y' = \frac{df}{dy}f(y)$$

From this we see two possible candidates for inflection points: Values for which $\frac{df}{dy} = 0$ and values for which $f(y)$. But as a matter of fact the latter values gave us already the equilibrium solutions, constant solutions, which have no inflection points. So the only meaningful possibility is at the points where $\frac{df}{dy} = 0$.
Again going back to the above example $y = y^2 - y$, And, $\frac{df}{dy} = 2y - 1$. Which means that solutions have points of inflections whenever their graphs reach the level $y = 1/2$.