Today we focus on the Existence and Uniqueness guarantee for a 1st order ODE: $y' = f(t, y)$ and separate it into two cases: linear and nonlinear.

Basically the guarantee tells us that, under certain assumptions, exactly one solution of an ODE exists. Of course, already from your brief experience in this course you know that 1st order ODEs have an arbitrary constant in their general solutions. Therefore, it is reasonable to expect ODE' to have exactly one solution only if we assume that the solution satisfies an initial value condition $y(t_0) = y_0$ in addition to the ODE.

The importance of having a uniqueness guarantee stems from the fact that the guarantee can be invoked without any knowledge about solving the ODE in terms of a specific formula. In fact, the existence guarantee will apply to situations where in fact no explicit formula is possible. Thus the existence that is in the guarantee is a "mathematical" existence as opposed to the existence of a formula for the solution.

Of course you have run into this sort of situation before when you studied the Fundamental Theorem of Calculus, which roughly says that an antiderivative exists for every continuous function. On the other hand, in Math 141 you encountered several important continuous functions that do not have antiderivatives expressible in terms of formulas.

We now turn to the first order linear case: $y' + py = g, \qquad y(t_0) = y_0$. If I is an open interval containing t_0, and if in addition p and g have no discontinuities in I, then a unique solution exists on the entire interval I. This is the end of the guarantee in the linear case!

We illustrate how to apply it with the following IVP: $(t^2 - t)y' = t - (t - 1)y, \qquad y(2) = 1$
Before we can see if the guarantee does apply we need to put the ODE into the required form $y' + py = g$. We then identify the p and g, and check out where their discontinuities are located. For this purpose we write:

$$y' + \frac{t-1}{t^2 - t} y = \frac{t}{t^2 - t} \qquad y(2) = 1$$

and rewrite it as:

$$y' + py = g \qquad y(2) = 1 \qquad \text{with} \qquad p = \frac{1}{t} \qquad \text{and} \qquad g = \frac{1}{t-1}$$

Obviously the discontinuities of p and g lumped together are at 0 and 1. Now we try to make the most out of the guarantee; so we take I to be the largest open interval containing $t_0 = 2$ but not any of the discontinuities. Obviously the choice is $I = (1, \infty)$. Note, that if the ODE were the same but t_0 would have been 1/3, then we would have chosen $I = (0, 1)$

It is important to realize that the actual value of y_0 did not enter into the consideration here. Therefore the visualization of the answer can be indicated in a 1-dimensional drawing of the real axis, with the discontinuities crossed out, the point t_0 marked and the interval I highlighted.

In the linear case the interval I does not depend on the choice of y_0 but the nonlinear case is much different. For example, we already saw that the IVP $y' = -t/y, y(0) = -y_0$ has the solution $y = -\sqrt{(y_0)^2 - t^2}$ which exists exactly on the interval $(-y_0, y_0)$ and no larger interval. Thus the I is very heavily dependent on y_0 in the nonlinear and hence the general guarantee of existence we cannot expect anything about the size of I.

The uniqueness part is also more problematic for nonlinear ODE's. Actually, you may be asking what good is the uniqueness guarantee? Is it not the case that "the more solutions the merrier"? Unfortunately, that answer is NO! Indeed we will see in this course several DE's for which we do not have a systematic procedure of locating all solutions. Instead we follow a hunch about what the solution might look like and eventually based on that hunch manage to locate a solution. Without a uniqueness guarantee we could not rely on that solution in modeling physical phenomena because without a systematic procedure for locating all solutions we cannot be sure that some one else's hunch will not lead to another solution with an entirely different behavior. In the absence of a systematic solution procedure, which is actually the case in many instances, the uniqueness guarantee for a DE is essential.

With this in mind consider the IVP

$$y' = \frac{3}{2} y^{1/3}, \qquad y(0) = 0$$

A quick application of the technique of separation of variables yields $y = (t + c)^{3/2}$ and setting $t = 0$ and $y = 0$ gives $c = 0$. We are not quite done yet because this solution does not even exist at $t = 0$. Recall that $t^{3/2}$ is not defined for negative t and does not have a derivative at 0. However, we can make it into one that does. Consider

$$\begin{aligned} y(t) &= t^{3/2} && \text{if} \quad t > 0 \\ y(t) &= 0 && \text{if} \quad t = 0 \\ y(t) &= 0 && \text{if} \quad t < 0 \end{aligned}$$

Clearly this function satisfies the ODE on each of the intervals $(-\infty, 0)$ and $(\infty, 0)$. Now at $t = 0$ one needs to look at two difference quotients separtely and take their one-sided limits and veryify that indeed the above function y is also differentiable at $t = 0$ and its derivative is 0 there.

Unfortunately there infinitely many solutions as well. So see why, recall that the direction field of an autonomous ODE can be drawn be moving a collection of arrows along a vertical left or right as long as their slopes are not disturbed. Since a solution to a differential equation is determined by the relationship between the arrows in a direction field and the slopes of the tangent to the graph of a function, for any autonomous ODE a given solution can be shifted either right or left any amount without disturbing the fact that it solves the ODE. In symbols this means that if $y(t)$ is a solution then so is $w(t) = y(t - c)$ for any fixed number c. For example we take the solution constructed above and set $w(t) = y(t - 1)$. Then $w(0) = y(0 - 1) = 0$ according to the definition of y given above. This means we have two solutions to the IVP. So the uniqueness guarantee fails miserably for this IVP. Even more perplexing is the fact that $f(t, y)$ is everywhere continuous in the above examples

With this preparation, we now look at the existence and uniqueness guarantee in the nonlinear case. Since y_0 must be taken into account, our visualization of the conditions under which the guarantee applies requires the entire ty-plane.

For the IVP problem

$$y' = f(t, y) \qquad\qquad y(t_0) = y_0$$

if there is an open rectangle in the ty-plane containing the point (t_0, y_0) but not any discontinuities of $f(t, y)$ and $f_y(t, y)$, then a unique solution of the IVP exists. This is the abrupt end of the guarantee in the nonlinear case! (And, nothing can be said about the size of the interval of existence of the solution, without further analysis.)

Recall that the meaning of the $f_y = \dfrac{\partial f}{\partial y}$ appearing in the guarantee is the partial derivative of f with respect to the variable y.

We illustrate this with the following IVP:

$$(1 - t)y' - (1 - y)^{1/3} = 1, \qquad\qquad y(0) = 1$$

To apply the guarantee to this nonlinear 1st order ODE's we first rewrite it:

$$(1 - t)y' = \frac{(1 - y)^{1/3}}{1 - t} + \frac{1}{1 - t} = f(t, y), \qquad\qquad y(0) = 1$$

We also find

$$f_y = \frac{1/3}{(1 - y)^{2/3}(1 - t)}$$

Clearly $f(t, y)$ is continuous everywhere in the ty-plane except along the vertical line $t = 1$. However, that is not the whole story. (From time to time you will run across examples where $f(t, y)$ does tell the whole story. However, this is not to give any justification for not looking at the discontinuities of f_y in each and every example.) f_y also has discontinuities along the horizontal line $y = 1$ and they must also be excluded in order to apply the guarantee. And at $(0, 1)$ the guarantee does not apply. However, there are many points remaining in the ty-plane where we can apply the guarantee.