Today we consider problems involving bodies falling under the influence of gravity and air resistance. Probably the latter force was not considered in your encounters with falling in previous courses. Of course, for some relatively streamlined objects like baseballs, gravity is the main force and air resistance can be neglected but for some objects, like feathers, neglecting air resistance destroys the essential features of the motion of the object.

The primary physical tool for the analysis of the motion is Newton's 2nd law which says that for an object moving under the influence of a certain force, its mass times its acceleration are equal to that force. If of course may be the case that the force changes as the object moves, as is the case when with air resistance is taken into account.

In these problems we will always use kilograms as the unit of mass, meters as the unit of length and seconds as the unit of time. We will also make the simplifying assumption that the acceleration of an object of mass 1 due to gravity g in a vacuum is -10 meters/sec^2, taking the upward direction to be positive. (This of course is an approximation. It is worthwhile because it allows us to focus on the essence of the problem which is difficult to do when the conventional -9.8 meter/sec^2 approximation is used.)

These problems are most easily dealt with in terms of one dependent variable y of the independent variable t (time), the derivative y' for which we also use the auxiliary variable v (velocity) and the second derivative y'' for which the auxiliary variable a is used.

So as mentioned above, in a vacuum, the acceleration of an object of mass m due to gravity is: $mv' = mg$. If we take air resistance into account we need to add (or perhaps subtract) on an additional force. A rough approximation of the force due to air resistance is obtained by assuming that it is proportional to speed and that it acts in a direction opposite to the velocity of the object. That is, if the object is moving upward, then air resistance works together with gravity to reduce its upward velocity And if the object is already falling, then air resistance works against gravity to increase its already negative velocity (reduce the speed at which it is falling). The constant of proportionality is roughly called the drag coefficient, k. So the ODE for velocity becomes

$$mv' = mg - kv$$

Please check that the sign in front of kv is correct when the object is moving upward,(i.e, when v is positive) and also when the object is moving downward, (i.e, when v is negative).

With this brief introduction we can solve a problems.

Suppose that an object of mass 2 kg is thrown upward from ground level with an initial velocity whose magnitude is 50m/s. Assume that air resistance exerts a force whose magnitude is four times the speed of the object. (Recall that speed is the magnitude of the velocity). The problem is to find the maximum height reached by the object and the time that the object returns to ground level?

We first write an IVP satisfied by v, the velocity of the object. Check that it is valid when the object is moving upward and when the object is moving downward.

The differential equation and initial condition which model v in this problem is:

$$2v' = -20 - 4v \qquad v(0) = 50$$

Divdiding through by 2 gives

$$v' = -10 - 2v \qquad v(0) = 50$$

Note that when When the object is moving upward v is positive and gravity and air resistance are both exerting a downward force on the object. When the object is moving downward v is negative and gravity is still exerting a downward force on the object but air resistance is pushing it upward.

We draw a direction field for this ODE in order to get an idea of what is happening to the velocity of the object. own? Click here for the Direction field Java applet and enter the given differential equation. for the direction field.

Note that $v = -5$ is an equilibrium solution. The point in time when $v = 0$ is when the object reaches the maximum height. Upto that point v decreases from $50m$/s to 0. After the object reaches its highest point the velocity v

becomes negative but can never go below the equilibrium solution which is -5m/s and hence 5 is a lower bound for the speed of the object on its way down. Thus the velocity is always between $50m$/s and -5/s Note that the amount of time it takes the velocity to reach its maximum will not be the amount of time the object takes to return to its starting position because its average speed on the return trip will be much lower that the average speed on the way up to its maximum height.

We now solve the ODE for v is both linear and separable. Our choice is to solve it as a linear equation (but the other choice is also good.)

$$1v' + 2v = -10 \qquad v(0) = 0$$

The integrating factor is $\mu = e^{t/2}$ and hence

$$ve^{2t} = -5e^{2t} + C$$

Plugging in the initial condition gives

$$C = 50 \qquad\qquad v = -5 + 55e^{-2t}$$

To find when the object returns to its starting position we need a formula for $y(t)$ the displacement of the object above the ground at time t. (Either use Newton's method to solve the equation $y(t) = 0$ or graph the function y using a computer/calculator to approximate the root.)

So this we integrate

$$y' = v = -5 + 55e^{-2t} \qquad y(0) = 100$$

So

$$y = -5t + \frac{55}{2}e^{-2t} - \frac{55}{2}$$

To find when the object hits gound we need to find the positive solution of the following equation

$$0 = -5t + \frac{55}{2}e^{-2t} - \frac{55}{2}$$

This equation can be solved using Newton's method, guessing that a positive solution is near $t = 11.5$. A method with limited accuracy, but adequate for this course, is graphing the function s and reading the cursor's t coordinate at the t-intercepts.

The time that the object took to reach its maximum height was $t = -\frac{1}{2}\ln(1/11) = 2\ln(11) \approx 4.8$seconds whereas the return trip takes about 6.7seconds, which confirms the prediction we made about the return trip from the maximum height taking substantially longer.