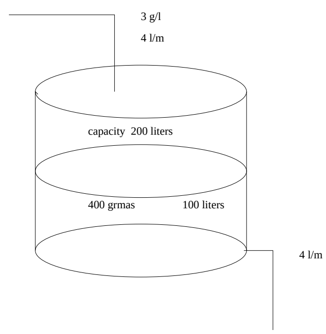
Today we resume our modeling of mixture problems. We change just one parameter which changes the nature of the problem substantially. Namely, today we assume that the rate at which mixture enters the tank is not equal to the rate at which well mixed mixture leaves the tank. The first consequence of this assumption is that the volume of the mixture in the tank varies. In case the volume is increasing we need to know the capacity of the tank because the experiment must be stopped when the tank fills completely.

To illustrate the process let us assume that a mixture with concentration of 3 g salt per liter water flows in at the rate of 2 l/min, and the capacity of the tank is 200 l. We keep the other parameters the same as they were yesterday; namely, the concentration of the mixture flowing into the tank is 3 g salt per liter water flows and it enters the tank at the rate of 4 l/min. Also, initially there are 400 g disolved in 100 liters of water in the tank.

Our goal is to answer the question: How much salt will there be in the tank at the moment it is completely filled?

We again arrange the data given above as follows:



Before solving the problem we consider two alternate problems that can be solved without any calculations in order to get a ball park figure for the answere to the question we were asked. If the tank were completely filled from empty with the incoming mixture, then the quantity of salt in the tank when full would clearly be 600 g. And if the outlet pipe were completely closed starting with a mixture 400 grams of salt dissolved in a 100 liters of water , would give 700 grams salt in the tank when completely full. Therefor a ballpark figure for the answer we are seeking is somewhere between 600g and 700g.

We now return to the question at hand. The analysis proceeds as before except that the volume is no longer is $V = 100$ but it is $V = 100 + 2t$ because with each minute that passes the volume is increased by 2 liters. From this we see that the tank is completely full when $t = 50$. Therefore we need a formula for $Q(t)$ and need to set $t = 50$ in that formula.

So, t he ODE for $Q(t)$ is

$$Q' = \frac{3\,\text{liter}}{\text{min}}\frac{4\,\text{g}}{\text{liter}} - \frac{4\text{liter}}{\text{min}}\frac{Q\,\text{g}}{100 + 2t\,\text{liter}}$$

So the ODE and initial condition are:

$$Q' = 12 - \frac{2Q}{100 + 2t} \qquad Q(0) = 400$$

Now the integrating factor becomes $\exp(\ln(50 + t)) = (50 + t)$ and we obtain:

$$((50 + t)Q)' = 12(50 + t)$$

And integration the above gives

$$(50 + t)Q = 6(50 + t)^2$$

The setting $t = 0$, $Q = 400$ gives $C = 50(400) - 6(50)^2 = 20(10)^3 - 15(10)^3 = 5(10)^3$ Therefore,

$$Q = 6(50 + t) + \frac{5(10)^3}{50 + t}$$

So,

$$Q(50) = 650$$

which is right in the middle of the ball park in which we expected it.

The integration could have been done slightly differently to give the following expression

$$(50 + t)Q = 600t + 6t^2 + D$$

Setting $t = 0$ and $Q(0) = 400$ in this give $D = 2(10)^4$. Then

$$Q = \frac{600t + 6t^2 + 2(10)^4}{50 + t} \qquad \text{and} \qquad Q(50) = \frac{600(50) + 6(50)^2 + 2(10)^4}{(10)^2} = \frac{3(10)^4 + 1.5(10)^4 + 2(10)^4}{(10)^2} = 650$$

which agrees with the previous calculation. The choice of which calculation to use is a matter of personal preference.