# Math 251 Sections 1,2,10 Lect Notes 8/31/09

A significant portion of this course is devoted to analyzing physical phenomena via DE models; this process is called mathematical modelling. We begin a week of modelling with a model for the balance in savings accounts and loans which accumulate interest and to which payments are made over a period of time.

Note that although the objectives of a savings account and a loan are distinct, from the view of modeling the balance in the account over a period of time, they can me modeled by the same differential equation. Let $P = P(t)$ be the balance in either one at time $t$ years. There are two causes for a change in $P$ over time: interest is calculated and added to the account and payments are made to the account. We will assume that both interest is credited to the account and constant payments are made continuously. The latter is a departure from physical reality but it allows us to come up with a very simple linear ODE for $P$. (Because, one cannot stand a the teller's window making a small payment each and every single second of even a few not hours, not to mention a few years.)

In the case of the savings the payments increase the balance; whereas, for in the case of the loan the payments reduce the balance $P$. In either case the constant rate of payments can be represented as $k$ dollars per year, with the understanding that in the first case $k$ is positive and in the second negative.

In both cases the crediting of interest increases $P$. And we now focus on finding an expression for the rate of change due to compounding the interest continuously and assume temporarily that $k$ is zero.

We introduce the notation $P_0$ for initial balance in an account and $r$ for the annual rate (a percent expressed as a decimal). If interest is credited only at the end once a year the balance at the end of the first year is $P(1) = P_0 + rP_0 = (1+r)P_0$. If the interest is compounded semiannually (i.e. paid into the account twice a year) then the balance at 1/2 year is $P(1/2) = (1+(r/2))P_0$ and if no withdrawls are made until the end of the year the balance at the end of the first year will be $P(1) = (1+(r/2))P(1/2) = (1+(r/2)^2P_0$ In this fashion we see that if there are $n$ equally spaced compounding periods in a year, then $P(1) = (1+(r/n)^nP_0$.

We now assume that interest is compounded continuously. This is impossible to achieve even with extremely fast computers but by taking the limit as $n \to \infty$ gives us a formula to represent this extreme situation:

$$P(1) = \lim_{n\to\infty} (1+(r/n))^n P_0$$

This limit is evaluated very easily. In fact in Math 140/141 you saw that

$$e = \lim_{m\to\infty} (1+(1/m))^m$$

The limit which the bank needs to evaluate can be converted to this one using the simple substitution $r/n = 1/m$. Then $n = rm$ and

$$P(1) = \lim_{n\to\infty} (1+(r/n))^n P_0 = \lim_{m\to\infty} (1+(1/m))^{rm} P_0 = \left(\lim_{m\to\infty} (1+(1/m))^m\right)^r P_0 = e^r P_0$$

From this we easily see that $P(2) = e^r P(1) = e^{2r} P_0$, $P(3) = e^r P(2) = e^{3r}_P 0$ and in general $P(t) = e^{rt} P_0$. That is the balance $P$ grows exponentially if the interest rate is compounded continuosly. By differentiation we arrive at the conclusion that

$$P'(t) = re^{rt} P_0 = rP(t)$$

Or, more briefly,

$$P' = rP$$

And if we now assume that $k$ is nonzero, then

$$P' = rP + k \qquad \text{with} \qquad P(0) = P_0$$

Now let's consider some problems related to this model.

Suppose a person opens an IRA (Retirement Savings Account) with an initial deposit of $\$10^3$. At what annual rate $k$ must money be deposited so that at the end of 30 years the balance in the account will be $10^6$. We assume that the annual interest rate of 10% is compounded continuously.

The ODE with initial condition that models this problem is:

$$P' = 0.1P + k, \qquad \text{and} \qquad P(0) = 10^3$$

we are also given that $P(30) = 10^6$ and we need to find $k$. We solve the above IVP using the solution metho for linear ODE's: The integrating factor for

$$P' - 0.1P = l$$

is $e^{-0.1t}$. Mutliplying through gives

$$(Pe^{-0.1t})' = ke^{-0.1t}$$

Integrating both sides with respect to $t$ gives:

$$Pe^{-0.1t} = -10ke^{-0.1t} + C$$

We can plug in the initial condition to find that $C = P_0 + 10k = 10^3 + 10k$ right away. I.e.,

$$Pe^{-0.1t} = -10ke^{-0.1t} + P_0 + 10k = -10ke^{-0.1t} + 0^3 + 10k$$

We now plug in $t = 3$ and $P = 10^6$ to determine $k$.

$$10^6e^{-3} = -10ke^{-3} + 10^3 + 10k$$

We can either use a calculator to solve for $k$ or we can use the approximation $e^3 \approx 20$ to solve for $k$ by hand. Using the latter we get

$$5(10^4) = -\frac{k}{2} + 10^3 + 10k$$

So

$$k = \frac{49(10^3)}{\frac{19}{2}} = 5157.90$$

So over a period of 30 years $1000 + 30 \times 5157.90 = 155736.84$ is put into the account but a $10^6$ dollars accumulates due to the compounding of the interest.

Loans are modeled by the same ODE except that the quantity $k$ representing the annual rate of payment is negative because in this case it reduces the rate of growth of the loan. (It is not advisable to change the ODE to reflect this fact by changing the plus sign to a minus sign).

So, at a 10% annual rate, what is the largest 30 year mortgage that is possible if the most one can afford to pay is $\$10^4$ per year.

The ODE and formula for $P$ in this problem are exactly as for the one above except now $k = -10^4$ , $P(30) = 0$ and it is $P_0$ that is unknown.

$$Pe^{-0.1t} = -10ke^{-0.1t} + P_0 + 10k = 10^5e^{-0.1t} + P_0 - 10^5$$

Finally we plug $t = 30$ and $P = 0$, to obtain the following equation of $P_0$

$$0 = 10^5e^{-3} + P_0 - 10^5 = \frac{10^5}{20} + P_0 - 10^5$$

So

$$P_0 = 10^5(1 - \frac{1}{20}) = \frac{19}{20}10^5 = 9.5 * 10^4$$