In this lesson we continue to solve more general linear ODE's and derive the formula for integrating factors that we stumbled upon in the last lesson for a rather special case. We begin by repeating the definition of linear ODE's for good measure.

**Definition** The 1st order ODE $y' = f(t, y)$ is called **linear** if the right hand side can be rewritten as follows:

$$y' = g(t) - p(t)y \qquad \text{or} \qquad y' = g - py$$

Here $g$ and $p$ are functions only of $t$ and either one may be a constant function including the constant function zero. (The minus sign in the above definition appears as a matter of convenience.)

A typical example of a linear ODE is: $y' = 3t + 2y$, and an example of a nonlinear is $y' = e^{t+y}$. Note that the first one is not separable whereas the second one is. Obviously, there are ODE's that are neither separable nor linear, eg, $y' = 3t + y^2$ and ODE's that are both linear and separable, eg, $y = 2y - 100$ .

Now consider a generic 1st order linear ODE: $y' = g - py$. We shall see that, in mathematical theory, it is possible to alter the ODE by multiplying it through by an "integrating factor" and obtain a very easily solved ODE, also in theory. For this purpose we first move the term $-py$ to the left hand side of the ODE and then we ask ourselves whether it is possible to multiply this equation by a function $\mu$ so that the left hand side is the derivative of a product of two functions. If the answer is yes, then we can integrate each side separately because there is nothing easier than integrating the derivative of a function and hopefully integrating the product sitting on the right hand side will not be too difficult either.

If we can find such a function $\mu$, then it is called an **integrating factor**.

So we now look for a formula for finding $\mu$. Let's write out precisely what we are looking for. We are faced with the ODE

$$y' + py = g$$

and we wish to find a function $\mu$ that has the property:

$$(y\mu)' = (y' + py)\mu$$

or equivalently

$$(y\mu)' = y'\mu + py\mu$$

On the other hand, the product rule applied to $(y\mu)'$ states that

$$(y\mu)' = y'\mu + y\mu'$$

By comparing the above two formulas we see that we need

$$py\mu = y\mu'$$

Since we are not interested in finding the constant solution $y = 0$ (because it is obviously a solution if and only if $g = 0$) we can divide both sides by $y$ to obtain:

$$p\mu = \mu'$$

We immediately recognize this as being a separable ODE for the unknown function $\mu$ of the independent variable $t$. This means that

$$\frac{\mu'}{\mu} = p$$

We integrate both sides with respect to $t$:

$$\ln \mu = \int p\, dt$$

and we then see that integrating factor $\mu$ is

$$\mu = e^{\int p\, dt}$$

(Remember this formula and/or its derivation.)

Why are we allowed to ignore the constant of integration in the process of finding $\mu$? Well we have the specific goal of finding just one integrating factor. (When we solve an ODE we seek a general solution, i.e., a solution containing an arbitrary constant which enables us to solve a variety of initial value problems.) Here need one $\mu$ that works and have absolutely no use for more! However, in the next step we do not have the freedom of ignoring constants of integration:

$$(y\mu)' = g\mu$$

$$y\mu = \int g\mu\, dt$$

$$y = \frac{1}{\mu} \int g\mu\, dt$$

Let's solve a few more linear ODE's using this procedure.

Consider the ODE:

$$y' = -\frac{2}{t} y + t^2 \cos t$$

This is a linear ODE even though the function multiplying $y$ is nonconstant. According to our formula for the integrating factor:

$$\mu = e^{-\int \frac{2}{t} dt} = e^{-2 \ln |t|} = e^{\ln t^{-2}} = \frac{1}{t^2}$$

Multiplying through by $\mu$ gives:

$$\left(\frac{1}{t^2} y\right)' = \frac{1}{t^2} t^2 \cos t + C = \cos t + C$$

Integrating both sides gives

$$\frac{1}{t^2} y = \sin t + C$$

And solving for $y$ is easy:

$$y = t^2 \sin t + Ct^2$$

Consider the following IVP:

$$(t-1)y' + 3y = t, \qquad y(0) = 1$$

To check that this ODE is linear and find the integrating factor, the coefficient of $y'$ must be equal to 1:

$$y' + \frac{3}{t-1} y = \frac{t}{t-1}$$

Now it is obvious that it is linear and that the integrating factor is:

$$\mu = e^{\int \frac{3}{t-1} dt} = e^{3 \ln(t-1)} = e^{\ln(t-1)^3} = (t-1)^3$$

Multiplying through by $\mu$ gives:

$$\big(y(t-1)^3\big)' = t(t-1)^2 = t^3 - 2t^2 + t$$

And integrating both sides gives:

$$y(t-1)^3 = \frac{1}{4}t^4 - \frac{2}{3}t^3 + \frac{1}{2}t^2 + C$$

Before trying to solve for $y$ we plug in $t = 0$ and $y = 1$ to find that $C = -1$ Therefore

$$y = \frac{3t^4 - 3t^3 + 6t - 1}{12(t-1)^3}$$