

We now move from separable ODE's to the 2nd of the three types of 1st order ODE's for which we will have explicit solution techniques in this course. Recall that we are assuming in this course that our 1st order ODE's can be rewritten in the form $y' = f(t, y)$. In general f can be a rather complicated function of t and y . We focus on a particularly simple f that does nothing more than multiply y by a function of t and also perhaps adds another function of t .

Definition The 1st order ODE $y' = f(t, y)$ is called **linear** if the right hand side can be rewritten as follows:

$$y' = g(t) - p(t)y \quad \text{or} \quad y' = g - py$$

Here g and p are functions only of t and either one may be a constant function including the constant function zero. (The minus sign in the above definition appears as a matter of convenience.)

Examples of linear ODE's are: $y' = y$, $y' = \cos(t)$, $y' = 4 - 2y$, $y' = e^t - 2y$, $y' = e^t + t^{-1}y$. Some of these examples are also separable. Examples of nonlinear ODE's are $y' = 1/y$, $y' = \cos(ty)$. In short an ODE is linear iff the only operations performed on y on the right hand side is multiplication by a function of t and/or addition of a function of t .

Applying term by term integration directly to a linear ODE does not lead to a solution; in fact it only leads to a dead end.

For example consider the ODE Consider the following ODE: $y' = e^{-t} - 3y$. We are very tempted to try to solve this ODE by integrating each term:

$$\int y' dt = \int e^{-t} dt - 3 \int y dt$$

Two of the three integrals above are easy to compute. The third one, however, is impossible since y is an unknown function of t and consequently its integral is just as unknown.

But shall see that it is possible to alter the ODE by multiplying it through by an "integrating factor" and obtain a very easily solved ODE, at least in theory. For this purpose we first move the term $-3y$ to the left hand side of the ODE:

$$y' + 3y = e^{-t}$$

and then we ask whether it is possible to multiply this equation by an exponential e^{at} so that the left hand side of

$$(y' + 3y)e^{at} = e^{-t}e^{at}$$

is the derivative of a product ye^{at} . If the answer is affirmative, then we can integrate each side separately:

$$\frac{d}{dt}(ye^{at}) = e^{at-t}$$

There is nothing easier than integrating the derivative of a function and integrating the exponential on the right is nearly as easy.

So let's look for e^{at} . What we are looking for is:

$$(y' + 3y)e^{at} = (ye^{at})'$$

We apply the distributive law to the left hand side and the product rule to the right hand side:

$$y'e^{at} + 3ye^{at} = y'e^{at} + aye^{at}$$

Obviously the equality of the two expressions above holds exactly when $a = 3$ and in this case our ODE is:

$$(ye^{3t})' = e^{2t}$$

Integrating both sides:

$$\int (ye^{3t})' dt = \int e^{2t} dt$$
$$ye^{3t} = \frac{1}{2}e^{2t} + C$$

Finally we obtain the following formula for y :

$$y = \frac{1}{2}e^{-t} + Ce^{-3t}$$

If we are given an initial condition, for example $y(0) = 1$, then the value of C can be found by simply setting $t = 0$ and $y = 1$ and solving for C in the above formula. One observation that ought to be made is that solutions of this ODE approach zero after a long time, regardless of their value at $t = 0$.

Let's solve some other ODE's using this procedure.

Consider the following ODE: $y' = t + 3y$ It is obviously a linear ODE and we rewrite it as

$$y' - 3y = t$$

By analogy to the above we see that an integrating factor is e^{-3t}

$$(ye^{-3t})' = e^{-3t}t$$

The integral of the right hand side requires integration by parts:

$$ye^{-3t} = \int e^{-3t}t dt = -\frac{1}{3}e^{-3t}t + \frac{1}{3} \int e^{-3t} dt = -\frac{1}{3}e^{-3t}t - \frac{1}{9}e^{-3t} + C$$

$$y = -\frac{1}{3}t - \frac{1}{9} + Ce^{3t}$$

As far as long time behavior goes, we note that if $y(0)$ is greater than $1/9$, then the solution tends to ∞ as $t \rightarrow \infty$.

Also, if $y(0) \leq -1/9$, then the solution approaches $-\infty$ as $t \rightarrow \infty$.

After after Maxima is used to solve an ODE, entering the the command

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method;
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asks it to reveal the method is used. For example,

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deqn: 'diff(y,t)-y^2=0 ;  
soln: ode2(deqn,y,t) ;  
method;
```

and

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deqn: 'diff(y,t)-y=0 ;  
soln: ode2(deqn,y,t) ;  
method;
```

Maxima's response is "separable" for the first and "linear" for the second.

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