We start this lesson with some terminology and then review the solution method used last to solve some simple 1st DE's called separable.

In the defintion of DE that was given last time the number of variables of the unknown function was left open. We attach two different names depending on how many variables it does have: We call it an Ordinary DE (ODE) if it it the function we are solving for has one variable and call it a Partial DE (PDE) if the unknown function has more than one.

Everyone is familiar with the usual notations for derivatives of a function of one variable with respect to its variable:y', $\frac{dy}{dt}$, y'', $\frac{d^2y}{dt^2}$,etc. Since Math 231 is not a prerequisite for this course we need to review the notations used for derivatives in case the function y has more than one variable, eg, $y = y(t, x)$, y is a function of t and x.

In this case $\frac{\partial y}{\partial t}$ and y_t are two notations for the derivative of y with respect to t. When this notation is used we assume that for the purpose of computing the derivative x is a constant. **It must not be interchanged with the notation** $\frac{dy}{dt}$ which would implicitly contain the assumption that also x is a function of t and hence x is not an independent variable. This is a DIFFERENT concept, called total derivative, and the chain rule for functions of two variables is needed to compute $\frac{dy}{dt}$.

Let's do some simple examples of computing partial derivatives:
$$y = t^2 + x^3 \qquad y_t = 2t \qquad \frac{\partial y}{\partial t} = 2t \qquad y_x = 3x^2 \qquad \frac{\partial y}{\partial x} = 3x^2$$

$$y = t^2 x^3 \qquad y_t = 2tx^3 \qquad \frac{\partial y}{\partial t} = 2tx^3 \qquad y_x = 3t^2 x^2 \qquad \frac{\partial y}{\partial x} = 3t^2 x^2$$

$$y = \sin(tx) \qquad y_t = \cos(tx)x \qquad \frac{\partial y}{\partial t} = \cos(tx)x \qquad y_x = \cos(tx)t \qquad \frac{\partial y}{\partial t} = \cos(tx)t$$

Now let's return to the main topic for today: solving a collection of ODE's called **separable** first order ODE's.

When we try to solve a first order ODE, we naturally think about finding the indefinite integral in order to eliminate the unknown derivative from the expression, and thereby obtain a formula involving y. Therefore we need to remember which indefinite integrals involving an unknown y can be evaluated, even thougth the integrand contains an unknown $y(t)$ Consider. e.g., the integrals $\int y \, dt$, $\int y^2 \, dt$, $\int sin(y) \, dt$, and contrast them with the following integrals: gives: $\int yy' \, dt = \frac{1}{2}y^2 + C$, $\int y^2 y' \, dt = \frac{1}{3}y^3 + C$, $\int \sin(y)y' \, dt = -\cos(y) + C$. The difference between the two collections is that each integrand in the second batch of integrals contains a factor of y' and hence it is a simple substitution that alllows us to replace the integral with a formula involving the unknown function y.

So let us return to the last ODE we looked at last time From the above we see that the ODE $P' = 2P - 100$. Or, using y as the unknown function $y' = 2y - 100$. Note that this ODE cannot be solved by term wise integration. Indeed that would lead us to $\int y \, dt$ from which we cannot remove the integral. But a simple "Divide and Conquer" (or more accurately "Divide and Integrate") strategy does work:

$$\frac{y'}{y - 50} = 2$$

Both sides of this equation can easily be integrated with respect to t:

$$\int \frac{y'}{y - 50} \, dt = \int 2 \, dt$$

$$\ln|y - 50| = 2t + C$$
$$|y - 50| = e^{2t+C} = e^C e^2 t$$
$$y - 50 = e^{2t+C} = \pm e^C e^2 t$$

Note that here e^C cannot be zero. On the other hand, we lost a solution $y = 50$ when we divided by $y - 50$. We therefore write

$$y - 50 = C_1 e^{2t}$$

where C_1 can be any real number including zero. We choose $C = 1$, if we need a solution satisfying $y(0) = 51$. $C = 0$ to satisfy $y(0) = 50$. and $C = -1$ to satisfy $y(0) = 50$. These choices indicate our conclusions based on the direction field of this ODE last time were correct.

For the answer we introduce the **Defintion** The ODE $y' = f(t, y)$ is called **separable** if $f(ty)$ can be replace by $g(t)h(y)$ where $g(t)$ is a function solely of t and $h(y)$ is a function solely of t.

And for separable ODE's we can hope to apply the "Divide and Integrate" technique:

Solve the initial value problem: $y' = -\frac{t}{y}$, $y(0) = -4$. What is the domain of the solution?

This leads us to the following definition: A first order ODE written as

$$y' = f(t, y)$$

is called separable if $f(t, y)$ can be factored into a product product of two functions $g(t)$ and $h(y)$, a function of t alone and a function of y alone.

The following three steps should be followed to solve separable ODE's.

    a. Divide both sides by $h(y)$.

    b. Integrate both sides with respect to t.

    c. Check for lost solutions.

As a first example consider $y' = -t/y$ together with the initial value problem $y(0) = -4$. This is a separable ODE because the right hand side is the product of a function of t: $g(t) = -t$ and a function of y: $h(y) = \frac{1}{y}$. We now integrate divide both sides by $h(y)$ (or simply multiplied by y) and integrate with respect to t: $\int yy' \, dt == \int t \, dt$ to obtain $\frac{1}{2}y^2 = -\frac{1}{2}t^2 + C$, or, equivalently, $y^2 = -t^2 + C_1$ or $y^2 + t^2 = C_1$ Plugging in $t = 0$, $y = -4$, gives $C = 16$ which we recognize as being the equation of a circle in the ty-plane. However, a circle is not the graph of function! In many situations in this course we will be very happy with an equation that cannot be solved for one of the variables in terms of the other. But since it this example it is so easy to solve lets do it: $y = -\sqrt{16 - t^2}$, where the minus sign is chosen to make it satisfy the initial condition. Also note that y is not differentiable at $t = \pm 4$ and hence cannot be said to solve the ODE there. There this solution exists on the interval $(-4, 4)$.

Let's try one more example:$y' + y^2 \sin(t) = 0$ $y(\pi/2) = 0$. Here in order to apply "Divide and Integrate" we need to more the 2nd term to the right hand side: $y = y^2 \sin(t)$ and divide by y^2 to obtain $\int y^{-2}y' \, dt == -\int \sin(t) \, dt$ which gives $\frac{-1}{y} = \cos(t) + C$, or equivalently However, we are unable to choose C so as to satisfy the given initial condition because a fraction can only be zero if the numerator is zero and $-1 \neq 0$. Did something go wrong here? NO! We simply forgot that a solution maybe excluded when we divide an equation. In this case the constant function $y = 0$ is a solution of the ODE but divding by y^2 excluded it. And before declaring that the ODE is solved we must include it with the formula that we found. Indeed $y = 0$ solves the given intial value problem. If the initial value problem were replaced by $y(\pi/2) = 1$ then the solution would be given by

$$y = \frac{1}{1 - \cos(t)}$$

Before finishing this lesson it is worthwhile to go over the procedure for getting Maxima to solve ODE's. Of course, we could just instruct Maxima to do the algebra and calculus steps that we would normally do by hand to solve the ODE. But that would not utilize all the algorithms preprogrammed into Maxima for solving 1st and 2nd order ODE's. The instructions that tell Maxima to select and apply the appropriate routine to solve an ODE is called ode2. Thus to instruct Maxima to solve the ODE $y' = -\frac{t}{y}$ with IVP $y(0) = -4$ we apply the following steps:

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deqn: 'diff(y,t)=-t/y ;
soln: ode2(deqn,y,t) ;
ivp: ic1(soln, t=0, y=-4) ;
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Please note that the apostrophe before the symbol diff is needed as it indicates to Maxima that the derivative cannot be evaluated (because y itself is unknown) and must dealt with as an unknown.