Math 251 Sections 1,2,10 Lect Notes 12/7/09

Today we explain how building blocks can be found for PDE's. This method is called **separation of variables**. This method cannot be used to solve all PDE's but its claim to fame is that it does give a procedure for discovering building block for these three very important PDE's, as well as many others.

Suppose the unknown function u in our PDE is a function of the independent variables x and t. The method is based on the assumption that the PDE has a solution which is expressible as a product of a function $X(x)$ of only x and $T(t)$ of only t:

$$u(x,t) = X(x)T(t)$$

To simplify the notation, today and only today, we allow ourselves the freedom of using the notation $X' = \frac{dX}{dx}$ together with $T' = \frac{dT}{dt}$. Such abuse of notation in other contexts carries a very heavy penalty.

To illustrate the method of separation of variables, let's use it to find $u(x,t)$ a solution to the following PDE

$$u_t = u_x + u$$

such that $u(x,0) = 3e^{4x}$.

For this purpose let us sssume that $u(x,t) = X(t)T(t)$. Then plugging into the equation we obtain

$$XT' = X'T + XT$$

We now try to separate the variables (only functions of one variable appearing on each side):

$$\frac{T'}{T} = \frac{X'}{X} + 1$$

Since a function of t is equal to a function of x in the above equation, both must be a constant; call it λ. Thus we obtain two equations, each one very easy to solve:

$$\frac{T'}{T} = \lambda \qquad \frac{X'}{X} + 1 = \lambda$$

Thus

$$T = d_2 e^{\lambda t} \qquad X = d_1 e^{(\lambda-1)x}$$

and finally

$$u(x,t) = d_3 e^{\lambda(t+x)-x}$$

Now in order to satisfy $u(x,0) = 3e^{4x}$ we plug into the above formula to match $d_3 e^{\lambda(\lambda x - x)} = 3e^{4x}$. This means we need to choose $d_3 = 3$ and $\lambda = 5$.

Now let us go back to the heat equation and see how one can discover the functions which we called "building blocks". That is we

$$u_{xx} = u_t$$

try to discover ALL the functions $f(x)$ with $0 < x < L$ for which it is possible to find a **NONZERO** solution $u(x,t) = X(x)T(t)$ for $t > 0$ satisfying the boundary conditions $u(x,0) = f(x)$ with $0 < x < L$ and $u(0,t) = u(L,t) = 0$ for any $t > 0$.

Assuming that $u(x,t) = X(x)T(t)$ we rewrite the PDE as follows:

$$X''(x)T(t) = X(x)T'(t)$$

We more everything involving X and or x to the left hand side and everything involving T and or t to the right hand side:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

We conclude that each side of the above formula is a constant, with respect to t and with respect to x. We denote this constant by $-\lambda$ (the minus sign is chosen in order to connect with the notation used last time and does not affect the mathematics here).

$$\frac{X''(x)}{X(x)} = -\lambda \qquad \text{and} \qquad \frac{T'(t)}{T(t)} = -\lambda$$

Now let us impose the boundary conditions

$$u(0,t) = u(L,t) = 0, t > 0, \quad u(x,0) = f(x)$$

where $f(x)$ is the initial temperature distribution function. The boundary value problem may be easy to solve for some $f(x)$ but not for others. Our analysis should reveal for which $f(x)$ it is easy to solve.

The first 2 conditions translate into the boundary value problem for $X(x)$:

$$X'' + \lambda X = 0 \qquad X(0) = 0, \quad X(L) = 0$$

But last time we found all eigenvalue eigenfunction pairs for this boundary value problem. They are

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \qquad X(x) = \sin\left(\frac{n\pi}{L}x\right)$$

where n is a positive integer. Solving the second equation $T' = -\lambda T$ is even easier: $T(t) = Ce^{-\lambda t}$. So if we choose $X(x) = f(x)$ to be one of the above eigenfunctions, then the boundary condition $u(x,0) = f(x)$ requires that we choose $C = 1$ in the solution. We conclude that

$$u(x,t) = X(x)T(t) = \sin\left(\frac{n\pi}{L}\right)e^{-\lambda t} = \sin\left(\frac{n\pi}{L}\right)e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

What happens if we change the imposed boundary conditions to the following:

$$u_x(0,t) = u_t(L,t) = 0, t > 0, \quad u(x,0) = f(x)$$

where $f(x)$ is the initial temperature distribution function?

The first 2 conditions translate into the boundary value problem for $X(x)$:

$$X'' + \lambda X = 0 \qquad X'(0) = 0, \quad X'(L) = 0$$

In the last FAQ we found all eigenvalue eigenfunction pairs for this boundary value problem. They are

$$\lambda = 0 \qquad X(x) = 1$$

and also

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \qquad X(x) = \cos\left(\frac{n\pi}{L}x\right)$$

where n is a positive integer. Solving the second equation $T' = -\lambda T$ is even easier: $T(t) = Ce^{-\lambda t}$. So if we choose $X(x) = f(x)$ to be one of the above eigenfunctions, then the boundary condition $u(x,0) = f(x)$ requires that we choose $C = 1$ in the solution. We conclude that

$$u(x,t) = X(x)T(t) = 1e^{\lambda t} = 1$$

in the case $\lambda = 0$ is the eigenvalue and for the remaining eigenvalues:

$$u(x,t) = X(x)T(t) = \cos\left(\frac{n\pi}{L}\right)e^{-\lambda t} = \cos\left(\frac{n\pi}{L}\right)e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

We finish with the observation that not every PDE can be solved by the separation of variables. For example consider

$$u_t + u_x = t$$

If we try to separate variables by setting $u(x,t) = X(x)T(t)$ then we get

$$XT' + X'T = t$$

But no matter how hard the variables just will not separate.