

Today we return to second order constant coefficient homogeneous linear ODE's and consider boundary value problems for them. A boundary value problem for an ODE asks for a solution on an interval, eg,  $[0, l]$  with it's value or the value of its derivative at each endpoint specified. The obvious difference between boundary value problem and IVP's for ODE's that requirements are imposed at two different points instead of specifying two different items at the same point. A much more subtle and more dramatic difference is that even for linear ODE's with constant coefficients the uniqueness guarantee fails and this is the aspect upon which we focus our attention today.

The techniques we develop today are applied to four examples in Chapter 10. We illustrate one of them and leave the remaining three to be worked on the FAQ's, quizzes, and final exam.

Today we use the symbol  $X$  to denote an unknown function of the independent variable  $x$ . Consider the the following boundary value problem:

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(2) = 0$$

Here  $\lambda$  is an arbitrary constant. It may be positive, negative. or zero. Obviously one solution is  $X = 0$ . However, the uniqueness guarantee may fail. We would like to determine the  $\lambda$  for which it does. Those values of  $\lambda$  are called **eigenvalues** and also a solution  $X$  that is not  $X = 0$ . That solution will be the **eigenfunction** associated with the eigenvalue.

Our search for eigenvalues is separated into three cases.

**First**, if  $\lambda < 0$  then setting  $\omega = \sqrt{-\lambda}$  gives the general solution is

$$X = c_1 \cosh(\omega x) + c_2 \sinh(\omega x)$$

Setting  $x = 0$  in  $X$  gives  $c_2 = 0$  and then setting  $x = 2$  in  $X$  gives  $\omega c_1 \cosh(\omega 2) = 0$  Sinc  $\cosh$  is never zero, we conclude that  $c_1$  must be zero. Thus there are no eigenvalues when  $\lambda < 0$ .

If  $\lambda = 0$  then the general solution is  $X = mt + b$ . Plugging in  $t = 0$  gives  $b = 0$  and plugging in  $t = 2$  gives  $m2 = 0$  which says that  $m = 0$ . Therefore,  $\lambda = 0$  is also not an eigenvalue.

If  $\lambda > 0$  then setting  $\omega = \sqrt{\lambda}$  the general solution is:

$$X = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

Setting  $t = 0$  in  $X$  gives  $c_1$ . We now set  $t = 2$  in  $X$  and we see that  $c_2 \sin(2\omega) = 0$ . In order to avoid having also  $c_2 = 0$  we need to have  $2\omega = n\pi$ ,  $n$  any nonnegative integer. Thus we obtain the following eigenvalues and eigenfunctions:

$$\lambda = \left(\frac{n\pi}{2}\right)^2 \quad \sin\left(\frac{n\pi}{2}x\right)$$

If 2 is replace by  $L$  then we obtain the following eigenvalues and eigenfunctions:

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \sin\left(\frac{n\pi}{L}x\right)$$

The eigenfunctions in this problem appeared in when we were finding the temperature of a rod with ends held in ice water.

2. Find all eigenvalues and eigenfunctions of the following boundary value problem:

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(2) = 0$$

**ANS.** If  $\lambda < 0$  then setting  $\omega = \sqrt{-\lambda}$  gives the general solution is

$$X = c_1 \cosh(\omega x) + c_2 \sinh(\omega x)$$

Then

$$X' = \omega(c_1 \sinh(\omega x) + c_2 \cosh(\omega x))$$

Setting  $x = 0$  in  $X'$  gives  $c_2 = 0$  and finally setting  $x = 2$  in  $X$  gives  $c_1 \cosh(\omega 2) = 0$ . Since  $\cosh$  is never zero, we conclude that  $c_1$  must be zero. Thus there are no eigenvalues when  $\lambda < 0$ .

If  $\lambda = 0$  then the general solution is  $X = mx + b$  and then setting  $x = 0$  in  $X'$  gives  $m = 0$  and setting  $x = 2$  in  $X$  gives  $b = 0$ . Therefore,  $\lambda = 0$  is not an eigenvalue.

If  $\lambda > 0$  then setting  $\omega = \sqrt{\lambda}$  the general solution is:

$$X = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

and

$$X' = \omega(-c_1 \sin(\omega x) + c_2 \cos(\omega x))$$

Setting  $t = 0$  in  $X'$  gives  $c_2 = 0$ . We now set  $t = 2$  in  $X$  and obtain that  $c_1 \cos(2\omega) = 0$ . In order to avoid having  $c_1 = 0$  we need to have  $2\omega = \pi(n + 1/2)$ ,  $n$  any nonnegative integer. Therefore we obtain the following eigenvalues and eigenfunctions:

$$\lambda = \left(\frac{\pi(2n+1)}{4}\right)^2 \quad \cos\left(\frac{\pi(2n+1)}{4}x\right)$$