Math 251 Sections 1,2,10 Lect Notes 12/3/09

Today we assume that the thin metal plate is circular, aka a disk, and consider the problem of finding its temperature after a long time under the assumption that the temperature at the boundary is given by a function F that does not vary with time.

It is convenient to assume that the radius of the disk is 1 and that its center is at the origin. It is also convenient to use polar coordinates for solving this problem. For additional convenience, we modify the definition of polar coordinates so that the angle θ has a range $(-\pi, \pi]$ instead of the usual range $[0, 2\pi)$. In this setting the function F which represents the temperature on the boundary can be viewed as a function of the angle θ on the half open interval $(-\pi, pi]$. Also, the temperature function can be viewed as a function of r and θ with r being in the interval $[0, 1]$. The boundary value problem that we need to solve is then given by the requirement $u(1, \theta) = F(\theta)$, for $-\pi < \theta \leq \pi$. But what is the PDE that we need to satisfy?

The Laplace equation in rectangular coordinates is $\Delta u = u_{xx} + u_{yy} = 0$. And it is obvious that in polar coordinates the Laplace equation is **NOT**: $u_{rr} + u_{\theta\theta} = 0$. For example, the function x is harmonic but in polar coordinates $x = r\cos\theta$ does not satisfy the above equation:

$$\frac{\partial^2 r\cos\theta}{\partial r^2} + \frac{\partial^2 r\cos\theta}{\partial\theta^2} \neq 0$$

In Math 231 the chain rule for functions of more that one variable is covered. Using this tool it can be seen that the correct expression for the Laplace equation in polar coordinates is

$$\frac{\partial}{\partial r}\left(r\frac{\partial r\cos\theta}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 r\cos\theta}{\partial\theta^2} = 0$$

Or in the simpler notation:

$$(ru_r)_r + \frac{1}{r}u_{\theta\theta} = 0$$

From this it is easy to see that not only $r\cos\theta$ and $r\sin\theta$ are harmonic but also for any positive n the functions $r^n\cos n\theta$ and $r^n\sin n\theta$ are harmonic. Let us verify this for the $r^n\cos n\theta$

$$\begin{aligned}
u_r &= nr^{n-1}\cos n\theta \\
ru_r &= nr^n\cos n\theta \\
(ru_r)_r &= n^2r^{n-1}\cos n\theta \\
u_\theta &= -nr^n\sin n\theta \\
u_{\theta\theta} &= -n^2r^n\cos n\theta \\
\frac{1}{r}u_{\theta\theta} &= -n^2r^{n-1}\cos n\theta \\
(ru_r)_r + \frac{1}{r}u_{\theta\theta} &= 0
\end{aligned}$$

We are now in a position to solve boundary value problem for the Laplace equation on the unit disk, the Dirichlet problem for the unit disk.

For example let's find the solution to the Dirichlet problem with boundary values

$$F(\theta) = 9 - 8\cos(7\theta) + 6\sin(5\theta)$$

on the boundary of the unit disk. That is, find the solution the of the Laplace equation on the unit disk $\{(r,\theta)|r<1\}$ which at the points $\{(1,\theta)|-\pi<\theta<\pi\}$ on the boundary of the unit disk has the property $u(1,\theta)=F(\theta)$.

A moments thought leads us to the the solution:

$$u(r,\theta)=9-8r^7\cos(7\theta)+6r^5\cos(5\theta)$$

This is far from being a realistic problem. In general if we are given a piecewise continuous $F(\theta)$, then the solution is:

$$u(r,\theta)=\frac{a_o}{2}+\sum_{n=1}^{\infty}r^n(a_n\cos(n\theta)+b_n\sin(n\theta))$$

where a_0, a_n, b_n are the Fourier coefficients of $F(\theta)$.

For example consider

$$F(\theta) = \begin{cases} 10, & \text{if } -\pi<\theta<0 \\ -10, & \text{if } 0<\theta<\pi \end{cases}$$

We easily find that $a_0=0$ since area below graph of an odd function is zero, as are all the other a_n. We find that

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{\pi}^{\pi} F(\theta)\sin n\theta\,d\theta \\ &= \frac{20}{\pi}\int_0^{\pi}\sin n\theta\,d\theta \\ &= \frac{-20}{n\pi}[\cos n\theta]_0^{\pi} \\ &= \frac{20}{n\pi}(1-\cos n\pi) \end{aligned}$$

$$u(r,\theta)=\sum_1^{\infty} r^n\frac{20}{n\pi}(1-\cos n\pi)\sin(n\theta))$$

We conclude with several observations. The value of the temperature at the center of the disk is always the average of the temperature on the boundary. If the value of the temperature on the boundary is a constant, then the temperature inside the disk is a constant (because all $a_n=0$, $b_n=0$ when $n\geq 1$). Finally if the value at the center of the disk is equal to the maximum on the boundary then the the temperature on the boundary is a constant and hence also on the entire disk (because the average can be equal to the maximum only if one is taking the average of constant values).

A somewhat more sophisticated fact is that the temperature is constant if the temperature at any point inside the disk reaches the maximum temperature on the boundary. All of these seemingly simple facts are related to subjects of great interest to pure and applied mathematicians.