Math 251 Sections 1,2,10 Lect Notes 12/2/09

We consider a thin rectangular metal plate that is insulated on top and bottom but along its edges which are held at temperatures that do not vary with time. But the temperature distributions may vary with respect to the position along the edges. We assume that the rectangle is $a$units wide and b units high and situated with its left lower corner at the origin of the xy-plane.

We only consider this problem after a long time. In this case the temperature has distributed itself as evenly as possible and which leaves no hot spots or cool spots inside the rectangle. An equation that describes this situation is the Laplace equation which says the Laplacian of u, denoted by Δu is zero inside the rectangle:

$$\Delta u = u_{xx} + u_{yy} = 0$$

Indeed, the Laplace equation says that a harmonic function, which is what solutions to the Laplace equation are called, has opposite concavity in directions that are at right angles to each other and since heat, like water, flows downhill, so to speak, it says that all the hot spots and cool spots have been diffused away.

We now need to search for building blocks of the Laplace and come up with the following as a possibility:

$$u(x, y) = e^{px} \sin(qy)$$

To see the relationship between p and q which makes u harmonic, we note that $u_{xx} = p^2 u$ and $u_{yy} = -q^2 u$. Therefore $u_{xx} + u_{yy} = (p^2 - q^2)u = 0$ if and only if $p^2 - q^2 = 0$. Therefore, we require $p = q$ of these building blocks.

However, in order to solve boundary value problems efficiently, it will be convenient to work with $\sinh(px) = \frac{e^{px} - e^{-px}}{2}$ known as the hyperbolic sine.

Every function $f(x)$ defined for all x can be written as the sum of an even function and an odd function because

$$f(x) = \frac{f(x) + f(-x) + f(x) - f(-x)}{2} = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

Note that the first fraction on the right is an even function and the second one is an odd function. Applying this little trick to write e^x as the sum of an even and an odd function leads us to $\cosh x$ and $\sinh x$. The reason why these are called hyperbolic trig functions has nothing to do with trigonometry; it is an acknowledgement that they satisfy many identities similar (but not identical) to the ones that the genuine trig functions satisfy. The one that concerns us here is that the derivative of $\sinh x$ is $\cosh x$ and the derivative of $\cosh x$ is $\sinh x$ (without a minus sign). Also observe that $\cosh x$ is never 0 and $sinhx$ is zero at $x = 0$. Since the derivative of $\sinh x$ is $\cosh x$, we see that $\sinh x$ is only zero once. This fact will be very helpful today and also later this week.

So on to solving a boundary value problem: Find the solution of the Laplace equation on the rectangle $\{(x, y) | 0 < x < 5, \quad 0 < y < 4\}$ which has the following values on the boundary:
$u(0, y) = 0$ if $0 < y < 4$
$u(x, 0) = 0$ if $0 < x < 5$
$u(x, 4) = 0$ if $0 < x < 5$
$u(5, y) = \sin(\frac{3\pi}{4} y)$ if $0 < y < 4$

The solution is:

$$u(x, y) = \frac{1}{\sinh(\frac{3\pi}{4} 5)} \sinh(\frac{3\pi}{4} x) \sin(\frac{3\pi}{4} y)$$

Note we simply choose q in our building block to match the given function on $x = 5, 0 < y < 4$. We then are left with no choice for p. However setting $x = 5$ in our building block does not give the $\sin(\frac{3\pi}{4}y)$ unless we place the building block by the constant $\dfrac{1}{\sinh(\frac{3\pi}{4}5)}$.

For a more realistic problem like $u(5, y) = g(y)$ with

$$g(y) \quad = \quad \begin{cases} 10, & \text{if} \quad 0 < y < 1 \\ 0, & \text{if} \quad 1 < y < 4 \end{cases}$$

we first need to find a sine series for $g(y)$ on $[0, 4]$:

$$\begin{aligned}
b_n &= \frac{1}{4}\int_{-4}^{4} g(y)\sin\left(\frac{n\pi}{4}y\right)\,dy \\
&= \frac{1}{2}\int_{0}^{4} g(y)\sin\left(\frac{n\pi}{4}y\right)\,dy \\
&= 5\int_{0}^{1} \sin\left(\frac{n\pi}{4}y\right)\,dy \\
&= -\frac{20}{n\pi}\left[\cos\left(\frac{n\pi}{4}y\right)\right]_0^1 \\
&= \frac{20}{n\pi}\left(1 - \cos\left(\frac{n\pi}{4}\right)\right) \\
g(y) &= \sum_{n=1}^{\infty} \frac{20}{n\pi}\left(1 - \cos\left(\frac{n\pi}{4}\right)\right)\sin\left(\frac{n\pi}{4}y\right)
\end{aligned}$$

And then

$$u(x, y) = \sum_{n=1}^{\infty} \frac{20}{n\pi}\left(1 - \cos\left(\frac{n\pi}{4}\right)\right)\frac{1}{\sinh(\frac{n\pi}{4}5)}\sinh(\frac{n\pi}{4}x)\sin\left(\frac{n\pi}{4}y\right)$$

Before we finish we need to make some additional observations. First, if the sides of the rectangle on which the given boundary value $g(y)$ are switched, the solution can be obtained by merely reversing the direction of the variable x from left to right:

$$u_{\text{left}}(x, y) = u_{\text{right}}(5 - x, y)$$

and if instead of specifying the boundary value on a vertical side of the rectangle it is specified on a horizontal side, the the problem is solved by merely interchanging x and y in the statement of the problem, then writing the solution as before, and then again interchanging x and y in the solution.

Finally, we note that any boundary value proble for a rectangle can be written as the sum of at most four problems each having the boundary value nonzero only on one of the sides of the rectangle.