Math 251 Sections 1,2,10 Lect Notes 11/30/09

Today we discuss the motion of a thin string (wire) with ends clamped under uniform tension. The PDE, which the displacement $u(x,t)$ of the string from its equilibrium position at x units from its left at time $t > 0$ satisfies, is called the wave equation. The motivation for its name will become apparent a little later. But we first make an observation that will make the wave equation plausible. Assume that at a fixed time t the shape of the string at several places is not straight. That is, it has concavity up or down of varying amounts. Our experience tells us that at the points where there is concavity there is also a force on the string that attempts to straighten the concavity and the magnitude of the force at a given point along the string is commensurate with the intensity of the concavity there.

This is the intuitiion behind the wave equation:

$$u_{tt} = a^2 u_{xx}$$

Here a^2 is a constant that depends on the tension as well as the material of the string.

We shall deal with two distinct boundary value problems. In the first we assume that the initial $u(x,0)$ displacement is a given function $f(x)$ and the initial velocity $u_t(x,0)$ of the string is zero. This is the situation when one considers the motion of a guitar string. And in the second we assume that the initial $u(x,0)$ displacement is zero function $f(x)$ and the initial velocity $u_t(x,0)$ of the string is a given function $g(x)$. This is the situation when one considers the motion of a piano string. The general problem of a vibrating string can be thought of as being simply a combination of these two; for this reason we do not need to go beyond these two.

A "building block": for the wave equation is:

$$u(x,t) = \sin(px)\cos(qt)$$

We see that $u_{tt} = -q^2 u(x,t)$ and $u_{xx} = -p^2 u(x,t)$. Therefore the wave equation is satisfied by the "building block" if $q = ap$.

We can now try our first problem. Suppose that a string of length 3 has its ends clamped so that $a^2 = 5$. If its initial displacement is $7\sin\left(\frac{4\pi}{3}x\right)$ and its initial velocity is zero, then what is its displacement at any $t > 0$.

The solution is simply:

$$u(x,t) = 7\sin\left(\frac{4\pi}{3}x\right)\cos\left(\sqrt{5}\frac{4\pi}{3}t\right)$$

Indeed, it satisfies the wave equation as well as the boundary conditions

$$u(0,t) = 0 \qquad u(3,t) = 0 \qquad u(x,0) = 7\sin\left(\frac{4\pi}{3}x\right) \qquad u_t(x,0) = 0$$

Note that for small $t > 0$ the shape of the string is just a constant (slightly less than 1) multiple of the original shape. Also note that by the sine of the sum (and difference) of two angles formula we can rewrite $u(x,t)$ as follows:

$$u(x,t) = \frac{7}{2}\left[\sin\left(\frac{4\pi}{3}x + \sqrt{5}\frac{4\pi}{3}t\right) + \sin\left(\frac{4\pi}{3}x - \sqrt{5}\frac{4\pi}{3}t\right)\right]$$

This expression contains the justification for the name wave equation: the solution is the sum of two waves (having the same shape as the initial displacement function) traveling in opposite directions with the same speed.

Our second problem deals with the same string except that its initial displacement is zero and its initial velocity is given by $4\sin\left(\frac{7\pi}{3}x\right)$.

It is obvious that the building block has to be changed to be

$$u(x,t) = \sin(px)\sin(qt)$$

with the same relationship between p and q: $p = aq$. If we were to write down the solution

$$u(x,t) = 4\sin\left(\frac{7\pi}{3}x\right)\sin\left(\sqrt{5}\frac{7\pi}{3}t\right)$$

then it would not be quite correct because it satisfies all the requirements except its initial velocity would be $u_t(x,0) = 4\sqrt{5}\sin\left(\frac{7\pi}{3}x\right)$, which differs from the requirement by a factor of $\sqrt{5}$. For this reason we take

$$u(x,t) = 4\frac{3}{7\pi\sqrt{5}}\sin\left(\frac{7\pi}{3}x\right)\sin\left(\sqrt{5}\frac{7\pi}{3}t\right)$$

Neither of the above problems were realistic. But having solved them leads us to conclude that we could solve realistic problems just as easily if we find the sine series for the initial displacement or initial velocity function on $[0, L]$.

One final comment about a realistic problem: the string of length 3 has an initial displacement $f(x) = 3/2 - |3 - x|$ and zero initial velocity. The initial shape of this string is a line with slope 1 between 0 and 3/2 and a line with slope -1 between 3/2 and 3. Here for a very small $t > 0$ the shape of the string is no longer just a multiple of the initial shape. Indeed according to the wave equation there is no force along the parts of the string consisting just of straight lines.

Much better intuition about the shape of the string comes from the representation of solution as

$$u(x,t) = \frac{1}{2}\left(f(x + \sqrt(5)t) + f(x - \sqrt(5)t)\right)$$

here we add two triangular shaped graphs and one shifted slightly to the right of 3/2 and the other slightly to the left 0f 3/2. Graphically, the sum is a trapezoid with the outer edges of the triangles and the combined bases forming three edges of the trapezoid and the top of the trapezoid formed by a line joining the "peaks" of the triangles.