Math 251 Sections 1,2,10 Lect Notes 11/19/09

Today we consider a rod that is insulated completely, also at its edges. We assume as before that f(x) represents its initial temperature distribution on [0, L] and we seek a formula for its temperature u(x, t) for any time t > 0.

Before we can make any progress towards this, we have to find a the mathematical expression that tells the heat equation that the ends of the rod under considering are insulated. For this purpose let us state that the feature of insulation that we wish to convey to the heat equation is that heat does not move to the left of zero. One way of insuring that heat does not move to the left of zero is to an identical rod with identical initial temperature distribution placed symmetrically at its left end. We can visualize the two rods as forming one rod of twice the length situated on the interval [-L, l] and having an even initial temperature distribution. From this we could conclude that for every fixed t > 0 the function u(x, t) is an even function of x.

An elementary property of even functions of x is that their derivatives are odd; in particular, their derivatives are zero at x = 0. So this is the property, required at x = 0 and by symmetry at x = L that is used to convey to the heat equation the fact that the ends of the rod are insulated.

Thus the boundary value problem that we will now consider for the heat equation is u(x, 0) = f(x), for $0 \le x \le L$, the given initial temperature distribution, $u_x(0,t) = 0$, and $u_x(L,t) = 0$, for t > 0.

Since the building block we have been using, $\sin(px)e^{qt}$ does not meet the requirements of the derivatives at the end being zero at the ends of the rod, we need to switch to the building block $\cos(px)e^{qt}$. The constraint on the value of q in terms of p remains the same as before: $q = -\alpha^2 p^2$. We can now turn to solve our first problem on insulated ends: Suppose the length of the insulated rod is L = 3cm and $\alpha^2 = 1.2$ and f(x), the temperature of the rod at time t = 0 is f(x) = 3. Also assume that the ends of the rod are insulated t > 0; ie, $u_x(0,t) = 0$, and $u_x(3,t) = 0$, for t > 0. Then what is u(x,t) for t > 0 and any 0 < x < L? Well this is a very simple problem. u(x,t) = 3 for any x and t. You could have answered this question without even coming to class or reading the text.

How about a more interesting question? The following initial temperature distribution is better: $f(x) = 3 + \cos\left(\frac{5\pi}{3}x\right)$. The solution is

$$u(x,t) = 3 + \cos\left(\frac{5\pi}{3}x\right)e^{-(1.2)^2(5\pi/3)^2t}$$

We should make two observations about the solution u(x, t). First, after a long time the solution is approximately 3.

Second, recall that the average of a function g(x) on an interval [a, b] is by definition:

$$\operatorname{average}_{[a,b]}(g(x)) = \frac{1}{b-a} \int_a^b \, g(x) \, dx$$

A simple calculation shows us that the average [0,L](u(x,t)) = 3. (If you don't see this immediately, then recall that the integral of a cosine function over the first half of its period is 0.)

But the above is far from being a realistic example. So consider the initial temperature distribution

$$f(x) = \begin{cases} 30 & \text{if } 0 \le x < 1\\ 0 & \text{if } 1 \le x < 3 \end{cases}$$

We need to find the cosine series of f(x). For this we use the even extension of f(x) to the full interval [-3,3].

$$a_{0} = \frac{1}{3} \int_{-3}^{3} f_{e}(x) dx = \frac{2}{3} \int_{0}^{1} 30 dx = 20$$

$$a_{n} = \frac{1}{3} \int_{-3}^{3} f_{e}(x) \cos\left(\frac{n\pi}{3}x\right) dx$$

$$= \frac{2}{3} \int_{0}^{3} f(x) \cos\left(\frac{n\pi}{3}x\right) dx$$

$$= \frac{2}{3} \int_{0}^{1} 30 \cos\left(\frac{n\pi}{3}x\right) dx$$

$$= \frac{20}{n\pi} \left[\sin\left(\frac{n\pi}{3}x\right)\right]_{0}^{1}$$

$$= \frac{20}{n\pi} \sin\left(\frac{n\pi}{3}\right)$$

So the cosine series is:

$$\frac{20}{2} + \sum_{n=1}^{\infty} \frac{60}{n\pi} \sin\left(\frac{n\pi}{3}\right) \cos\left(\frac{n\pi}{3}x\right)$$

Finally, the solution is:

$$10 + \sum_{n=1}^{\infty} \frac{60}{n\pi} \sin\left(\frac{n\pi}{3}\right) \cos\left(\frac{n\pi}{3}x\right) e^{-1.2(n\pi/3)^2 t}$$

Before we quit we again return to the long time behavior. The temperature at any x approaches 10 as $t \to \infty$. That is $\frac{a_0}{2}$. This quantity is the average temperature of the rod at any time, which corresponds to our intuition that a totally insulated rod approaches a constant which is its average temperature at $t \to \infty$

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