Math 251 Sections 1,2,10 Lect Notes 11/18/09

Today we consider a slight variation of the boundary value problem for the heat equation which we considered yesterday. Instead of assuming that the ends of the thin metal rod are kept in ice water, we will assume that the ends are kept at a constant temperature.

Before trying to solve this problem for a general initial temperature distribution $f(x)$, we first ask for what initial temperature distribution is the solution to this new sort of problem extremely simple?

Let's make the question more specific by stating the various parameters explicitly. Ie, we assume $\alpha^2 = 1.9$, $L = 4$, the temperature at the left end is constantly 0^oC, $u(0,t) = 0$, and the temperature at the right end is constantly 100^oC, $u(4,t) = 0$.

So, what initial temperature distribution, call it $g(x)$ leads to an extremely simple solution? Well, we use the same intuition argument that led us to the heat equation. The function $g(x)$ should go from 0 to 100 as x varies over the entire length of the rod but if it would do so without any concavity then also u_t would be zero. That means, that if $g(x) = 25x$, then $u(x,t)$ can be taken to be independent of t. But in that case $u(x,t) = 25x$.

This solution is called the steady state solution.

Now suppose the we change the initial temperature distribution to $f(x) = 50^o$C, assuming the same constant temperatures at the ends as before. In this case we have the idea that the problem can be broken into two parts: one the has a steady state solution and the other that has a transient solution (a solution that can be ignored for all practical purposes after a long time.)

We can either draw a couple of sketches of the domains of these solutions or we can complete the following tabel

	Init Temp	Temp at $x=0$	Temp at $x=L$
This Problem	50	0	100
=			
Steady State	25x	0	100
+			
Transient	50-25x	0	0

We see that our problem is solved by $u(x,t) = u_{\text{steady\_state}}(x,t) + u_{\text{transient}}(x,t)$ We already observed that $u_{\text{steady\_state}}(x,t) = 25x$ and if we find the sine series $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{4}x\right)$ for the initial temperature $50-25x$ on $[0,4]$ then the transient solution is

$$u_{\text{transient}}(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{4}x\right) e^{-1.9(n\pi/4)^2 t}$$

Frequently, we need to know the solution after a long time. In that case we do not need to actually find the sine series above. For example, if need to find the temperature of the rod at $x = 3.21$ after a long time, then we simply evaluate the steady state solution there and say $u(3.21,t) \approx 80.25^o$C for large t.