

We are now in a position to solve some boundary value problems for partial differential equations. Second order linear partial differential equations naturally fall into three categories: parabolic, hyperbolic and elliptic. We focus on an important representative of each category, the heat equation, the wave equation and the Laplace equation, and solve two or three different sorts of boundary value problems for each.

The first equation we shall study is the heat equation. The typical physical problem that leads to the heat equation involves predicting the temperature of a thin metal rod of length L , which is entirely insulated except possibly at its ends, based on a given initial temperature distribution $f(x)$ at a point x cm from its left end. It is convenient to assume that temperature is measured in degrees centigrade and a variety of devices are possibly placed at the ends of the rod that effect the temperature there.

The function $u(x, t)$ having two variables has a graph that resides in three dimensional space. We shall view cross sections of the this graph at various points in time. We shall also view the domain of the temperature distribution functions $u(x, t)$ as an infinite rectangle in the xt -plane: $\{(x, y) \mid 0 \leq x \leq L, 0 \leq t\}$. The name boundary value problem comes from the fact that the behavior of $u(x, t)$ is specified on the three portions of the boundary and our task is to find $u(x, t)$ for any (x, t) that lies inside the infinite rectangle.

We shall not give a formal derivation of the heat equation. Your textbook gives one but that is best left to a physics course where a full discussion of transfer thermal energy through matter can be conducted. However, we shall try to make the PDE that we will claim represents the temperature distribution $u(x, t)$ of the rod at x units from the left end of the rod at time t at least somewhat plausible. For this purpose visualize the *graph* of $u(x, t)$ for a fixed value of t . This graph has its ups and downs reflecting hot and cool spots. From our every day experience with heated objects we know that if for a given x the temperature at (x, t) is higher than that for nearby x then as time progresses that temperature will drop and also if for a given x the temperature at (x, t) is lower than that for nearby x then as time progresses that temperature will increase there. Now at hot spots one we see that $u_{xx}(x, t) < 0$ and at cool spots $u_{xx}(x, t) > 0$ (remember the second derivative test for local maxima and minima!) This combined with our intuitive feeling that a increase in temperature will occur at cool spots and and decrease in temperature will occur at hot sports leads us to conjecture that u_{xx} and u_t are some how connected. And the heat equation says that indeed they are connected by a constant that depends only on the material the object is made off known as **thermal diffusivity**

$$u_t = \alpha^2 u_{xx}$$

Before the end of this course we will see how we can discover some “building blocks” for this PDE (ie, solutions which are a product two very simple function). For the time being however let us assume that such “building blocks” can be found in some location, eg, a textbook or a building supply store. So let us assume that a $u(x, t) = \sin(px)e^{qt}$ is such a “building block”. Then let us see how p and q must be related and what sort of boundary value problems can be solved.

The relationship between p and q is easy to see. Just plug into the PDE: $u_t = qu$ and $u_{xx} = -p^2u$. Therefore: $u_t = \alpha^2 u_{xx}$ holds iff (if and only if) $q = -\alpha^2 p^2$.

Now what about a boundary value problem for the PDE? Well suppose the length of the insulated rod is $L = 3$ cm and $\alpha^2 = 1.9$ and $f(x)$, the temperature of the rod at time $t = 0$ is $f(x) = \sin(\frac{5\pi}{3}x)$. Also assume that the ends of the rod are held in ice water for $t > 0$. Then what is $u(x, t)$ for $t > 0$ and any $0 < x < L$?

Upon a moments reflection we see the solution to this problem is:

$$u(x, t) = \sin\left(\frac{5\pi}{3}x\right)e^{-(1.9)^2(5\pi/3)^2t}$$

It is that simple: $u(x, t)$ satisfies the PDE and also matches the conditions imposed on the three portions of the boundary of its domain:

$$u(x, 0) = f(x), u(0, t) = 0, u(3, t) = 0$$

The initial temperature distribution $f(x) = 4 \sin(\frac{5\pi}{3}x) + 6 \sin(\frac{7\pi}{3}x)$ would appear to be a slightly more complicated problem. However, not more than a couple of moments would be required to write its solution:

$$u(x, t) = 4 \sin(\frac{5\pi}{3}x) e^{-(1.9)^2(5\pi/3)^2 t} + 6 \sin(\frac{7\pi}{3}x) e^{-(1.9)^2(7\pi/3)^2 t}$$

Here, of course, we are relying on the fact that the superposition principle is valid for the heat equation: a linear combination of solutions of the PDE is also a solution.

One cannot help but feel that the above two examples with their rather specialized initial temperature functions $f(x)$ were contrived for presentation to a math class but are rather unlikely to occur in reality. This feeling is absolutely correct! Now here is where Fourier series are applied. Suppose that the initial temperature function is a function that has a sine series on the interval $[0, 3]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{3}x)$$

For example, suppose that $f(x) = 40^\circ\text{C}$ on $[0, 3]$. And we continue to suppose that the ends of the insulated rod are kept in ice water at $t > 0$. Then the solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{3}x) e^{-(1.9)^2(n\pi/3)^2 t}$$