Math 251 Sections 1,2,10 Lect Notes 11/13/09

Before we are able to deal with applications of Fourier series to solving boundary value problem for PDE's we need to address two questions:

1. How can we find a Fourier series for a funtion $f(x)$ that is only defined on the interval $[0, L]$?

2. How can we find a Fourier series for a function that consists only of sine terms or only of cosine terms?

Fortunately answers to these two questions are only required of us when they are presented simulataneously. In this situation the answer is rather simple.

If we wish to only see sine terms in the Fourier series (called sine series for short), then the function whose Fourier series we are calculating must be odd on $[-L, L]$. Consequently, if a funtion $f(x)$ is only defined on the interval $[0, L]$, then we compute the Fourier series of its odd extenstion $f_o(x)$ to $[-L, L]$. And similarly if we wish to have a cosine series for the function and it is only defined on the interval $[0, L]$ initially, then we compute the Fourier series of its even extenstion $f_o(x)$ to $[-L, L]$.

Your textbook gives new formulas for computing sine and cosine series, namely, formulas (7) and (8) on paages 596-7. These are entirely unnecessary and increase the liklihood of errors on quizzes and exams. Knowing the originaly formulas well is the best strategy.

We illustrate with the following example:

Consider the function

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x < 1 \\ 0 & \text{if} \quad 1 \leq x < 2 \end{cases}$$

Find a sine series for this function on interval $[0, 2]$. What is $\lim_{n\to\infty} s_n(-1)$? What is $\lim_{n\to\infty} s_n(-2)$?

The sine series is the Fourier series of the odd extension $f_o(x)$ $f(x)$ to the full interval $[-2, 2]$. Fortunately we do not need to come up with a formula for $f(x)$. Since the Fourier series of $f_o(x)$ on $[-2, 2]$ contains only sine terms we only need to find b_n which have even integrands:

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f_o(x) \sin\left(\frac{n\pi}{2}x\right)\, dx \\ &= \frac{2}{2}\int_{0}^{2} f(x) \sin\left(\frac{n\pi}{2}x\right)\, dx \\ &= \int_{0}^{1} 1 \sin\left(\frac{n\pi}{2}x\right)\, dx \\ &= -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}x\right)\right]_0^1 = -\frac{2}{n\pi}(\cos\left(\frac{n\pi}{2}\right) - 1) \end{aligned}$$

Therefore the sine series is:

$$\sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right) \sin\left(\frac{n\pi}{2}x\right)$$

If f_o is defined as required by the Fourier Convergence Theorem then we obtain $\lim_{n\to\infty} s_n(-1) = -1/2$ and $\lim_{n\to\infty} s_n(-2) = 0$.

Now find a cosine series for the same function on interval $[0,2]$. What is $\lim_{n\to\infty} s_n(-1)$? What is $\lim_{n\to\infty} s_n(-2)$?

The cosine series is the Fourier series of the even extension $f_e(x)$ $f(x)$ to the full interval $[-2,2]$. Fortunately we do not need to come up with a formula for $f(x)$. Since the Fourier series of $f_e(x)$ on $[-2,2]$ contains only cosine terms we only need to find a_0 and a_n which have even integrands.

To find a_0 we only need to visualize $\frac{1}{L}$ times the area below the graph of f_e from $[-2,2]$. The nonzero part of the area is a rectangle with base $[-1,1]$ and height 2. Its area is 2 and therefore $a_0 =$

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f_e(x)\cos\left(\frac{n\pi}{2}x\right)\,dx \\
&= \frac{2}{2}\int_{0}^{2} f(x)\cos\left(\frac{n\pi}{2}x\right)\,dx \\
&= \int_{0}^{1} 1\cos\left(\frac{n\pi}{2}x\right)\,dx \\
&= \frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{2}x\right)\right]_0^1 = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

Therefore the cosine series is:

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}x\right)$$

If f_e is defined as required by the Fourier Convergence Theorem then we obtain $\lim_{n\to\infty} s_n(-1) = 1/2$ and $\lim_{n\to\infty} s_n(-2) = 0$.

At this point in time you may be a little concerned by the fact that we have just found two different Fourier series for the same function f on $[0,2]$. To overcome this concern remember that we were working with two different functions on $[-2,2]$, namely f_e and f_o.