Math 251 Sections 1,2,10   Lec Notes   11/11/09

Several times we encountered hints that the sequence of partial sums $\{s_n(x)\}$ of the Fourier series of function $f(x)$ on the interval $[-2,2]$ does not converge to $f(x)$. This is certainly puzzling. After all did we not derive the following formulas

$$a_0 = \frac{1}{L}\int_L^L f(x)\,dx \tag{1}$$

$$a_n = \frac{1}{L}\int_L^L f(x)\cos\left(\frac{n\pi}{L}x\right)\,dx \tag{2}$$

$$b_n = \frac{1}{L}\int_L^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx \tag{3}$$

for the Fourier coefficients under the assumption that the partial sums $s_n(x)$ did converege to $f(x)$? That of course is what we did!

The puzzlement here stems from confusing a statement with its converse. We list both of them here in order to make the distinction:

A. If the sequence of partial sums $\{s_n(x)\}$ of the Fourier series of a piecewise continuous function $f(x)$ converege to $f(x)$ on $[-L,l]$, then the Fourier coefficients, a_0, a_n, and b_n, are given by formulas (1), (2), and (3).

B. If a piecewise continuous $f(x)$ is given on $[-L,L]$ and trigonometric polynomial $s_n(x)$ are constructed according to (1) (2) and (3), then the sequence $\{s_n(x)\}$ converges to $f(x)$ on $[-L,L]$.

What we have verified is Statement **A.**. But it is not logically equivalent to Statement **B.** and, in fact, as we suspected, Statement **B.** is not true without additional assumptions!

The Fourier Convergence Theorem tells us under what circumstances Statement **B.** is true.

i. $f(x)$ and $f'(x)$ are piecewise continuous on $(-\infty,\infty)$
ii. $f(x)$ is extended to be periodic with period $2L$ on $(-\infty,\infty)$
iii. At points of discontinuity x_0 of $f(x)$, $f(x_0)$ is the average of the one-sided limits of f at x_0.

Under the above assumptions the Fourier Convergence Theorem states that:

$$\lim_{n\to\infty} s_n(x) = f(x) \qquad \text{for} \quad \text{all} \qquad x$$

So for example if $s_n(x)$ are the partial sums of the Fourier series of $f(x) = xu(x)$ on the interval $[-2,2]$, then we can find the following limits $\lim_{n\to\infty} s_n(3/4)$ $\lim_{n\to\infty} s_n(2)$ $\lim_{n\to\infty} s_n(3)$ $\lim_{n\to\infty} s_n(43)$ by insuring that $f(x)$ satisfies the hypotheses, **i.**, **ii.**, and **iii.**, of the Fourier Convergence Theorem.

Indeed $f(3/4) = 3/4$, $f(2) = 1$ is the average of the one-sided limits because $x = 2$ is a point of discontinuity, and $f(3) = f(43) = 1$, since $f(x)$ has period $2L = 4$.

There is another issue related to the convergence of $s_n(x)$ to $f(x)$. That is, if f has a discontinuity at a point x_0, then it is not possible for $s_n(x)$ converge uniformly to $f(x)$ near x_0. However, the Yale University mathematical physicist Josiah Willard Gibbs discovered rate of nonuniformity known as Gibbs' phenomenon. Specifically, the discovery is that in any interval containing the discontinuity x_0, $s_n(x_0$ both overshoots and and undershoots $f(x)$ by approximately 8% of the total jump at x_0.