Math 251 Sections 1,2,10 Lect Notes 11/09/09

We will use the independent variable x here because in most of our applications the variable of the function $f(x)$ for we seek a Fourier series represents position and not time. But before define the Fourier series of a piecewise continuous function $f(x)$ on the interval $[-L, L]$ you need to review properties of integrals mentioned on the attached list of definite integration formulas for trig functions.

The integral of p.w. continuous function $f(x)$ on $[-L, L]$ is the sum of the integrals of this function over intervals of continuity. The Fourier series of $f(x)$ on $[-L, L]$ is:

$$\frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

where a_0, $a_n, b_n, n = 1, 2, 3, ...$ are sequences of constants called Fourier coefficients. It is instructive to write out a few terms in the above infinite series explicitly:

$$\frac{a_0}{2} + a_1 \cos\left(\frac{1\pi}{L}x\right) + b_1 \sin\left(\frac{1\pi}{L}x\right) + a_2 \cos\left(\frac{2\pi}{L}x\right) + b_2 \sin\left(\frac{2\pi}{L}x\right) + a_3 \cos\left(\frac{3\pi}{L}x\right) + b_3 \sin\left(\frac{3\pi}{L}x\right) + ...$$

A number of comments are in order. First, the constant term in the Fourier is $\frac{a_0}{2}$ and **not** a_0. Secondly, it is conventional to write sine before cosine. Thirdly, each term in the Fourier series can have a somewhat different period. However, a common period for the entire Fourier series is $2L$. Fourthly, just writing the definition of the Fourier series down, does not give us any knowledge about whether or not the Fourier series converges and if it does converge whether or not it converges to the original $f(x)$. This issue will have to be dealt with. However, one can already sense there is an issue here because $f(x)$ might be a nonperiodic function whereas the Fourier series has a period that depends on the choice L.

For the time being let us assume that in some sense it does converge to the original function $f(x)$ and try to find formulas for the Fourier coefficients: a_0, a_n, b_n.

For this purpose we find the definite integral on $[-L, L]$ under the assumptions that **I.** $f(x)$ is equal to its Fourier series and **II.** the order of summing an infinite series and performing a definite integration can be interchanged:

$$\begin{aligned}
\int_{-L}^{L} f(x)\,dx &= \int_{-L}^{L} \frac{a_0}{2}\,dx + \int_{-L}^{L} \sum_{1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\,dx \\
&= \int_{-L}^{L} \frac{a_0}{2}\,dx + \sum_{1}^{\infty} \int_{-L}^{L} a_n \cos\left(\frac{n\pi}{L}x\right)\,dx + \int_{-L}^{L} b_n \sin\left(\frac{n\pi}{L}x\right)\,dx
\end{aligned}$$

Note that because sine is odd all the integrals involving sine are zero and also from formula on the Trig Integrals handout (which is attached) all integrals involving cosine are zero. We conclude that

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} \frac{a_0}{2} = a_0 L\,dx \quad \text{consequently} \quad a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

Having a formula for a_0 gives us hope of being able to continue and find a formula for a_n. For this purpose we multiply the above assumption for $f(x)$ through by $\cos\left(\frac{m\pi}{L}x\right)$ and integrate from $-L$ to L, again assuming that the order of summation and integration can be interchanged:

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)\,dx$$
$$= \int_{-L}^{L} \frac{a_0}{2}\cos\left(\frac{m\pi}{L}x\right)\,dx + \int_{-L}^{L}\sum_{1}^{\infty} a_n\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right) + b_n\cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\,dx$$
$$= \int_{-L}^{L} \frac{a_0}{2}\cos\left(\frac{m\pi}{L}x\right)\,dx + \sum_{1}^{\infty}\int_{-L}^{L} a_n\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\,dx + \int_{-L}^{L} b_n\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\,dx$$

The integral involving $\cos\left(\frac{m\pi}{L}x\right)$ is zero by Formula (1). The integrals involving $\cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)$ are all zero because that product is an odd function. The integrals involving $\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)$ are all zero except for the case where $n = m$ when it is equal to L by Formula (4). Therefore, we obtain

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)\,dx$$

If one swithches multiplication by $\cos\left(\frac{m\pi}{L}x\right)\,dx$ for multiplication by $\sin\left(\frac{m\pi}{L}x\right)\,dx$ then one obtains

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)\,dx$$

Although we do not yet know what connection the Fourier series has with the original function, it is obvious that a given $f(x)$ together with a given interval $[-L, L]$ can have only one Fourier series.

Therefore, a function which is given by formula that is already a Fourier series requires no additional computation of its Fourier series. To illustrate this consider the following problem.

One of the following functions has a Fourier series on $[-12, 12]$ that is very easy to find and the other one is a bit more difficult. Find the very easy one:

$$f(x) = 4 + 5\cos\left(\frac{3\pi}{6}x\right) + 7\sin\left(\frac{6\pi}{24}x\right) \qquad g(x) = \frac{8}{9} + 10\cos\left(\frac{4\pi}{12}x\right) + 10\sin\left(\frac{5\pi}{24}x\right)$$

In fact, for $f(x)$ we easily see that the Fourier coefficients are $a_0 = 8$, $a_{12} = 5$, and $b_3 = 7$, and all other coefficients are zero. Whereas, for $g(x)$ the expression $\left(\frac{5\pi}{24}x\right)$ needs to be put into the form $\left(\frac{n\pi}{12}x\right)$ for some integer n and that fails because $\frac{5}{2}$ is not an integer.

Now let us consider a slightly more difficult problem. Let us find the Fourier on the interval $[-2, 2]$ of the Heaviside function $f(x) = u(x)$

We find

$$a_0 = \frac{1}{2}\int_{2}^{2} u(x)\,dx = \frac{1}{2}\int_{0}^{2} 1\,dx = 1$$
$$a_n = \frac{1}{2}\int_{2}^{2} u(x)\cos\left(\frac{n\pi}{2}x\right)\,dx = \frac{1}{2}\int_{0}^{2} 1\cos\left(\frac{n\pi}{2}x\right)\,dx = 0$$

remembering to use Forumula (1) to evaluate the last integral.

$$b_n = \frac{1}{2}\int_{2}^{2} u(x)\sin\left(\frac{n\pi}{2}x\right)\,dx = \frac{1}{2}\int_{0}^{2} 1\sin\left(\frac{n\pi}{2}x\right)\,dx = \frac{1}{2}\frac{2}{n\pi}\left[-\cos\left(\frac{n\pi}{2}x\right)\right]_0^2 = \frac{1}{n\pi}\left(1-\cos(n\pi)\right)$$

Note that all the a_n are zero as are all the b_n for even n. If maybe helpful to write out a few nonzero Fourier coefficients b_n: $b_1 = \frac{2}{\pi}$, $b_3 = \frac{2}{3\pi}$,$b_5 = \frac{2}{5\pi}$, ...

Finally the Fourier series is: $\frac{1}{2} + \sum_1^\infty \frac{1}{n\pi}(1-\cos n\pi)\sin\left(\frac{n\pi}{2}x\right)$

Derivation of Some Definite Integrals Involving Trig Functions

We start with a little warm up exercise. If we integrate $\sin\left(\frac{n\pi}{L}x\right)$ from $-L$ to L, then we get zero because sine is odd. However cosine is even and the interval $[0, L]$ is not symmetric. Nevertheless, we have the following formula valid for any n:

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{L}{n\pi}\left[\sin\left(\frac{n\pi}{L}x\right)\right]_0^L = 0$$

For somewhat more complicated formulas which are at the heart of Fourier series computations we use the following well known addition formulas:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad\qquad \sin(A-B) = \sin A\cos B - \cos A\sin B$$

Adding and dividing gives

$$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \tag{1}$$

Also,

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad\qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$$

Adding and dividing gives

$$\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \tag{2}$$

We will use the symbols m and n to denote nonnegative integers. The integrals involving trig functions are as follows:

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\,dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ L & \text{if} \quad m = n \end{cases} \tag{3}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\,dx = 0 \quad \text{for} \quad \text{any} \quad m, \quad n \tag{4}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\,dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ L & \text{if} \quad m = n \end{cases} \tag{5}$$

Since sine is an odd function and cosine is even, their product is odd. The integral of an odd function over a symmetric interval is zero. These simple observations establish formula (4). We will now derive formulas (3). The derivation of formula (5), which is analogous, is given in the textbook.

The derivation of formula (3) is based on formula (2) which expresses the product of two cosine function as the sum of another pair of cosine fucntion. That is, applying formula (2) with $A = \left(\frac{m\pi}{L}x\right)$ and $B = \left(\frac{n\pi}{L}x\right)$ gives the following:

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m+n)\pi}{L}x\right) + \cos\left(\frac{(m-n)\pi}{L}x\right)\,dx$$

If m and n are different nonnegative integers, then the integrals on the right hand side can easily be evaluated:

$$\begin{aligned} &\frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m+n)\pi}{L}x\right) + \cos\left(\frac{(m-n)\pi}{L}x\right)\,dx \\ &= \left[\frac{L}{2(m+n)\pi}\sin\left(\frac{(m+n)\pi}{L}x\right) + \frac{L}{2(m+n)\pi}\sin\left(\frac{(m-n)\pi}{L}x\right)\right]_{-L}^{L} \end{aligned}$$

The evaluation gives zero because the sine function is 0 at any integer multiple of π

Finally, If m and n are equal nonnegative integers, then the integrals on the right hand side can be evaluated just as easily.

$$\frac{1}{2}\int_{-L}^{L} \cos\left(\frac{2n\pi}{L}x\right) + \cos\left(\frac{0\pi}{L}x\right)\,dx = \left[\frac{L}{2n\pi}\sin\left(\frac{2n\pi}{L}x\right)\right]_{-L}^{L} + \frac{1}{2}\int_{-L}^{L} 1\,dx = 0 + \frac{1}{2}2L = L$$