

Today we consider a damped pendulum. Usually a pendulum is restricted swinging back and forth just a few degrees. Our pendulum however can pivot full revolutions in either direction. It simply needs to be started with sufficient initial velocity to carry it over top. Of course the pendulum is at rest (in equilibrium) when it is at bottom dead center.

We denote by x the angular displacement from equilibrium of the pendulum with mass m concentrated at the end of an arm of length L and with damping constant c . The 2nd order **nonlinear** homogeneous ODE for x is well-known:

$$x'' + \gamma x' + \omega^2 \sin x = 0 \quad \text{where} \quad \gamma = \frac{c}{mL} \quad \text{and} \quad \omega^2 = \frac{g}{L}$$

The motion of the pendulum can be analyzed by converting the second order ODE into a 2×2 system and sketching a phase portrait.

For this purpose set $y = x'$ (this reverses our previous convention when the unknown function y appeared in a second order ODE.)

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 \sin x - \gamma y \end{aligned}$$

We seek critical points by setting the right hand sides equal to zero. The first equation says that $y = 0$ at any critical point. From which we see that $\sin x$ must be zero at critical points. Therefore, the critical points are

$$\begin{pmatrix} n\pi \\ 0 \end{pmatrix} \quad \text{where} \quad n = 0, \pm 1, \pm 2, \pm 3, \dots$$

Because of the periodicity we only need to consider the critical points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

We now proceed with the linearization. In order to clarify the procedure let us assume that $\gamma = \omega^2 = 1$.

The linearization at the first critical point is

$$\begin{aligned} x' &= y \\ y' &= -\sin x - y \approx -x - y \end{aligned}$$

The matrix is $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ which is an asymptotically stable spiral whereas at the second critical point we introduce new coords $\mathbf{u} = \mathbf{x} - \begin{pmatrix} \pi \\ 0 \end{pmatrix}$ and find the linearization:

$$\begin{aligned} u' &= v \\ v' &= -\sin(u + \pi) - v = \sin u - v \approx u - v \end{aligned}$$

where $u = x - \pi$ and $v = y$. The matrix for the linearization is $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$. The characteristic equation is $r^2 + r - 1 = 0$ and the eigenvalues are $\frac{-1 \pm \sqrt{5}}{2}$. Since they have opposite sign, the second critical point is an unstable saddle.

These correspond to our intuition about a damped pendulum that a small displacement and velocity cause the pendulum to move with decaying oscillations. Whereas, it is very difficult (perhaps impossible in practice) to find a displacement and velocity that will cause the pendulum to eventually stop at a displacement of π .

It is instructive to check these predictions with Professor Mansfield's Phase Portrait Java applet