The procedure for approximating a nonlinear system by linear systems near each of its critical points is as follows: Locate the critical points. Near each one of the plug the new coords into the nonlinear system, drop the nonlinear terms, and draw the phase portrait near the critical point.

We illustrate this for the following system:

$$\begin{aligned} x' &= 2y - xy \\ y' &= x - \frac{3}{2}x^2 + y^2 \end{aligned}$$

We start by finding the critical points. From the first equation we see that we have two sorts of critical points $\begin{pmatrix} 2 \\ ? \end{pmatrix}$ or $\begin{pmatrix} ? \\ 0 \end{pmatrix}$ And from the second equation we see that if $x = 2$ then we must have $y^2 = 4$ and if $y = 0$ we are left with $x - \frac{3}{2}x^2 = 0$, ie, $x = 0$ or $x = \frac{2}{3}$ and which givess the following 4 critical possibilities: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

At $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the linearization is $\begin{aligned} x' &= 2y \\ y' &= x \end{aligned}$ whose matrix is $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. So the critical point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an unstable saddle.

At $\mathbf{x}_0 = \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$ we set $\begin{aligned} u &= x - 2/3 \\ v &= y \end{aligned}$ and $\begin{aligned} x &= u + 2/3 \\ y &= v \end{aligned}$ Therefore the linearized ODE at this point is

$$\begin{aligned} u' &= 2v - (u + \tfrac{2}{3})v & &\approx \tfrac{4}{3}v \\ v' &= (u + 2/3) - \tfrac{3}{2}(u + 2/3)^2 + v^2 & &\approx -u \end{aligned}$$

The matrix for this is $\begin{pmatrix} 0 & 4/3 \\ -1 & 0 \end{pmatrix}$. So the critical point at $\begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$ is a center.

At $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ we set $\begin{aligned} u &= x - 2 \\ v &= y - 2 \end{aligned}$ and $\begin{aligned} x &= u + 2 \\ y &= v + 2 \end{aligned}$ Therefore the linearized ODE at this point is

$$\begin{aligned} u' &= 2(v + 2) - (u + 2)(v + 2) & &\approx -2u \\ v' &= (u + 2) - \tfrac{3}{2}(u + 2)^2 + (v + 2)^2 & &\approx -5u + 4v \end{aligned}$$

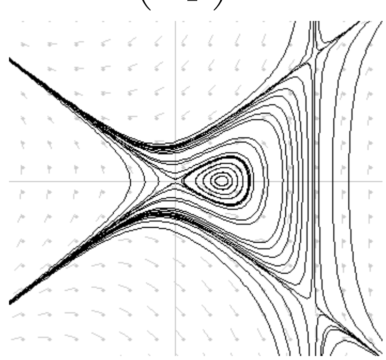
The matrix for this is $\begin{pmatrix} -2 & 0 \\ -5 & 4 \end{pmatrix}$. So the critical point at $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is an unstable saddle.

At $\mathbf{x}_0 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ we set $\begin{aligned} u &= x - 2 \\ v &= y + 2 \end{aligned}$ and $\begin{aligned} x &= u + 2 \\ y &= v - 2 \end{aligned}$ Therefore the linearized ODE at this point is

$$\begin{aligned} u' &= 2(v - 2) - (u + 2)(v - 2) & &\approx 2u \\ v' &= (u + 2) - \tfrac{3}{2}(u + 2)^2 + (v - 2)^2 & &\approx -5u - 4v \end{aligned}$$

The matrix for this is $\begin{pmatrix} 2 & 0 \\ -5 & -4 \end{pmatrix}$. So the critical point at $\mathbf{x}_0 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ is also an unstable saddle.

We combine all this into the phase portrait:



We now consider the Predator-Prey system:

$$
\begin{aligned}
x' &= 2x - xy \\
y' &= -3y + xy
\end{aligned}
$$

Setting

$$
\begin{aligned}
0 &= 2x - xy = x(2-y) \\
0 &= -3y + xy = y(-3+x)
\end{aligned}
$$

gives two critical points $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{x}_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

At $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the linearization is $\begin{matrix} x' = 2x \\ y = -3y \end{matrix}$ which has eigenvalues $r_1 = 2$ and $r_2 = -3$ which have opposite sign. Hence the origin is a saddle for for the linearized system.

At $\mathbf{x}_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ we set $\begin{matrix} u = x - 3 \\ v = y - 2 \end{matrix} \qquad \begin{matrix} x = u + 3 \\ y = v + 2 \end{matrix}$. Substituting this into the original equation gives

$$
\begin{aligned}
u' &= (u+3)(-v) \approx -3v \\
v' &= (v+2)u \approx 2u
\end{aligned}
$$

The eigenvalues of this system are purely imaginary. Thus this critical point is a center for the linearized system.