Today we begin the transition to nonlinear systems of ODE's by dealing with the nonhomogeneous linear case. The strategy we pursue is for a nonhomogoeneous linear system is to locate its critical point and introduce a new coordinate system which has its origin at this critical point. It turns out that system expressed in these coordinates is homogeneous. And at this point we analyze the system as we did before.

So let's fix notations: $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ and suppose that $\mathbf{x}_0$ is the critical point; ie, $A\mathbf{x}_0 + \mathbf{b} = \mathbf{0}$. Then we set $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$, placing the origin of the new coordinate system at $\mathbf{x}_0$. We now seek a formula for the system in terms of the new coordinate system $\mathbf{u}$. Note that

$$\mathbf{u}' = \mathbf{x}'$$

and also

$$A\mathbf{u} = A(\mathbf{x} - \mathbf{x}_0) = A\mathbf{x} - A\mathbf{x}_0 = A\mathbf{x} + \mathbf{b}$$

Therefore

$$\mathbf{u}' = A\mathbf{u}$$

To illustrate these concepts let us analyze the system:

$$\mathbf{x}' = \left(\begin{array}{cc} -2 & 1 \\ -5 & 4 \end{array} \right) \mathbf{x} + \left(\begin{array}{c} 1 \\ -2 \end{array} \right)$$

The critical point is determined by the equations $\begin{array}{l} -2x +1y + 1 = 0 \\ -5x \quad 4y - 2 = 0 \end{array}$ Subtracting 4 times the first row from the second gives $3x + 0y - 6 = 0$ That is, $x = 2$ and $y = 3$. We therefore set $\mathbf{u} = \mathbf{x} - \left(\begin{array}{c} 2 \\ 3 \end{array} \right)$. In terms of components this is $u = x - 2$ and $v = y - 3$. We need to analyze the homogeneous linear system $\mathbf{u}' = \left(\begin{array}{cc} -2 & 1 \\ -5 & 4 \end{array} \right) \mathbf{u}$. The characteristic equation of A is

$$0 = \det(A - rI) = \left| \begin{array}{cc} -2 - r & 1 \\ -5 & 4 - r \end{array} \right| = r^2 - 2r - 8 - (-5) = (r - 3)(r + 1)$$

which has two real eigenvalues $r_1 = 3$, $r_2 = -1$. Now

$$\mathbf{0} = (A - 3I)\xi = \left(\begin{array}{cc} -5 & 1 \\ -5 & 1 \end{array} \right) \xi \quad \text{gives} \quad \xi_1 = \left(\begin{array}{c} 1 \\ 5 \end{array} \right)$$

as the first eigenvector. Also,

$$\mathbf{0} = (A + I)\xi = \left(\begin{array}{cc} -1 & 1 \\ -5 & 5 \end{array} \right) \xi \quad \text{gives} \quad \xi_2 = \left(\begin{array}{c} 1 \\ 1 \end{array} \right)$$

as the second eigenvector. Therefore the general solution is:

$$\mathbf{u} = c_1 e^{3t} \left(\begin{array}{c} 1 \\ 5 \end{array} \right) + c_2 e^{-t} \left(\begin{array}{c} 1 \\ 1 \end{array} \right)$$

and in terms of the original coordinate system:

$$\mathbf{x} = c_1 e^{3t} \left(\begin{array}{c} 1 \\ 5 \end{array} \right) + c_2 e^{-t} \left(\begin{array}{c} 1 \\ 1 \end{array} \right) + \left(\begin{array}{c} 2 \\ 3 \end{array} \right)$$

The next two times we use the ideas of linear systems to approximate the phase portraits of nonlinear autonomous 2×2 systems.