Math 251 Sections 1,2,10 Lect Notes 10/29/09

Today we continue with the next possibility for the characteristic equation of a 2×2 autonomous linear homogeneous systems $\mathbf{x}' = A\mathbf{x}$: namely repeated eigenvalues. This situation itself splits into 2 cases as we shall see from the ensuing examples. Consider the system $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x}$ The characteristic equation is $(2-r)^2 = 0$ and we see that indeed in this case there is only one eigenvalue $r_1 = 2$. It is called a repeated eigenvalue. When we seek the corresponding eigenvector we find that it must satisfy

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\xi = \mathbf{0}$$

We see that any nonzero vector can be chosen as the eigenvector. So one possibility is to pick our favorite pair of nonmultiple vectors and express the general solution as:

$$\mathbf{x} = e^{2t}\left(c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Another possibility is to postpone writing down a general solution until we encounter an IVP, eg, $\mathbf{x}(0) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ and then tailor our general solution to the given IVP:

$$\mathbf{x} = e^{2t}\left(c_1\begin{pmatrix} \beta \\ \alpha \end{pmatrix} + c_2\xi_2\right)$$

where ξ_2 is any nonzero vector that is not a multiple of $\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$. The reason for this trickery is that it gives a very simple formula for the solution to the IVP that we encounter: $\mathbf{x} = e^{2t}\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$. We see that trajectory is a ray pointing towards the origin passing through the point $\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$. This means that all the trajectories of this system are rays. This critical point is called a **proper node**. It is unstable or asymptotically stable according as the single eigenvalue is positive or negative. The phase portrait is extremely easy to draw. Moreover, if the phase portrait of any other linear system contains more than four rays, then the only possibility that remains for it is to be a proper node as we shall see after we have covered all the six possibilities.

Another distinctive feature of the phase portrait is that all only two trajectories are parallel for large values of t. In the other cases corresponding to real eigenvalues all trajectories are parallel to for large t and all trajectories are parallel for very negative t.

For the second example we consider the system $\mathbf{x}' = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x}$ Again the characteristic equation is $r^2 + 2r + 1 = 0$ there is only one repeated eigenvalue $r_1 = -1$. When we seek eigenvectors but we only find one:

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\xi = \mathbf{0} \qquad \text{ie,} \qquad \xi_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

One solution is $e^{r_1 t}\mathbf{x}_1$ but where is the second non multiple solution coming from? We recall from our experience with 2nd order linear homogeneous ODE's that multiplication by t is helpful. This is a good start to solving our dilema but it requires a bit of modification. The reason for this complication becomes obvious when one thinks about the connection between (2×2) systems and 2ndd order linear homogeneous ODE's. Specifically that in the vector $\mathbf{x} = myvecxy$ the first component x is the derivative of the second component y. So if the second solution y_2 is merely t times the first solution y_1, then it is not the case that y_2' is t times y_1' because that would violate the product rule: $y_2' = y_2 + ty_2'$.

$$\begin{aligned} \text{If} \quad \mathbf{x} &= te^{r_1 t}\xi_1, \\ \text{then} \quad \mathbf{x}' &= \left(e^{r_1 t} + tr_1 e^{r_1 t}\right)\xi_1 \\ \text{whereas} \quad A\mathbf{x} &= te^{r_1 t}A\xi_1 = tr_1 e^{r_1 t}\xi_1 \end{aligned}$$

We see that there is a discrepency and to correct this we assume that

$$\begin{aligned} \text{If} \quad \mathbf{x} &= e^{r_1 t}\eta + te^{r_1 t}\xi_1 \\ \text{Then} \quad \mathbf{x}' &= r_1 e^{r_1 t}\eta + \left(e^{r_1 t} + tr_1 e^{r_1 t}\right)\xi_1 \\ \text{whereas} \quad A\mathbf{x} &= e^{r_1 t}A\eta + te^{r_1 t}A\xi_1 = e^{r_1 t}A\eta + tr_1 e^{r_1 t}\xi_1 \end{aligned}$$

Therefore, in order to have a solution we need:

$$r_1 e^{r_1 t}\eta + e^{r_1 t}\xi_1 = e^{r_1 t}A\eta \quad \text{or} \quad \text{equivalently} \quad (A - r_1 I)\,\eta = \xi_1$$

The vector η is called a **generalized eigenvector**
Let us continue with the example above and find η such that

$$(A - (-1)I)\,\eta = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

This means that

$$\left(\begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

That is

$$\begin{aligned} -\eta_1 - \eta_2 &= -1 \\ \eta_1 + \eta_2 &= 1 \end{aligned}$$

This will always be very easy to solve; for example take $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
So the 2nd solution we seek is

$$\mathbf{x}_2 = e^{-t}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + te^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and the general solution is

$$\mathbf{x} = e^{-t}\left(c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)\right)$$

It is easy to see that the second order ODE $y" + 2y' + y = 0$ is equivalent to the system we just have solved. Recall that the general solution we found for the second order ODE was $y = e^{-t}\,(c_1 + c_2 t)$. Since the conversion to a system involves the definition $y' = x$ we see that the solution to the system requires us to write y and $x = y' = e^{-t}\,(-c_1 - c_2 t + c_2)$ as a column vector

$$\begin{aligned} \begin{matrix} x = e^{-t}(& -c_1 & -c_2 t & +c_2 &) \\ y = e^{-t}(& c_1 & +c_2 t & +0 &) \end{matrix} &= e^{-t}\left(\begin{pmatrix} -c_1 \\ c_1 \end{pmatrix} + \left(\begin{pmatrix} c_2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -c_2 \\ c_2 \end{pmatrix}\right)\right) \\ &= e^{-t}\left(c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)\right) \end{aligned}$$

For this system the phase portrait is rather distinctive in that all trajectories eventually are parallel to the same direction both for very large t and for very negative t. The critical point this time is called an **improper node**. It is unstable or asymptotically stable according as the single eigenvalue is positive or negative. The phase portrait is roughly Figure II on the FAQ for 10/30/08.