Today we look at the situation where linear 2×2 systems has complex eigenvalues. We illustrate the general procedure in this case by dealing with three examples and begin with the following:

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix} \mathbf{x}$$

The characteristic equation is $r^2 + 16 = 0$ which has purely imaginary eigenvalues $\pm 4i$. The procedure we follow is entirely analogous to the the 2nd order linear homogeneous constant coefficient case; namely we found a complex solution and used its real and imaginary parts to produce the general solution.

For this purpose we form $A - 4iI = \begin{pmatrix} -4i & 2 \\ -8 & -4i \end{pmatrix}$ and seek a complex eigenvector. Note that the second row is equal to the first row multiplied by $-2i$. This means we have not made an error at this point in time. So the complex eigenvector is $\begin{pmatrix} 2 \\ 4i \end{pmatrix}$ or perhaps more neatly $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$ and a complex solution is

$$\mathbf{x}_d = e^{-4it} \begin{pmatrix} 1 \\ 2i \end{pmatrix} = (\cos 4t + i \sin 4t) \begin{pmatrix} 1 \\ 2i \end{pmatrix} = \begin{pmatrix} \cos 4t + i \sin 4t \\ 2i \cos 4t - 2 \sin 4t \end{pmatrix}$$

$$\mathbf{x}_1 = \text{Re}\, \mathbf{x}_d = \begin{pmatrix} \cos 4t \\ -2 \sin 4t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \text{Im}\, \mathbf{x}_d = \begin{pmatrix} \sin 4t \\ 2 \cos 4t \end{pmatrix}$$

And the general solution is $\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2$.

As far as the drawing trajectories, there are no rays among the trajectories of this system. As a matter of fact all the trajectories are periodic with period equal to $\frac{\pi}{2}$. Moreover, it is easy to see that each of $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy the equation of an ellipse: $4x^2 + y^2 = 4$. Therefore, every trajectory is an ellipse with the same eccentricity but different major axis. The only question is the direction. One way of figuring out the direction is to plot two nearby points, eg, $x(0)$ and $x(\pi/8)$. However, there is even an easier way. Just use the system itself to give the tangent to the trajectory at one point, for example the tangent to the trajectory at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is

$$\begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

] This is a vertical vector pointing upward at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which indicates that the ellipses are oriented counterclockwise.

The name of the critical point at the origin is **center**. Every trajectory that starts close to the origin stays close to the origin. However, it does not approach the origin (unless it is the origin). This situation is called stable. Among all the six different critical points a linear system can have at the origin this is the only one with this property. That is, a center is the only critical point that can correctly be called stable.

The two other types of complex eigenvalues can now easily be dealt with now. For example consider the following:

$$\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -8 & -1 \end{pmatrix} \mathbf{x}$$

The characteristic equation is $(r+1)^2 + 16 = 0$ which has complex eigenvalues $-1 \pm 4i$. An eigenvector can be found by looking at the matrix $A - (-1 + 4i)I = \begin{pmatrix} -4i & 2 \\ -8 & -4i \end{pmatrix}$, which is the same as before. Therefore, we use the same complex eigenvector $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$ and a complex solution is:

$$\mathbf{x}_d = e^{(-1+4i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} = e^{-t} (\cos 4t + i \sin 4t) \begin{pmatrix} 1 \\ 2i \end{pmatrix} = e^{-t} \begin{pmatrix} \cos 4t + i \sin 4t \\ 2i \cos 4t - 2 \sin 4t \end{pmatrix}$$

And, as before we find

$$\mathbf{x}_1 = \text{Re}\,\mathbf{x}_d = e^{-t}\begin{pmatrix} \cos 4t \\ -2\sin 4t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \text{Im}\,\mathbf{x}_d = e^{-t}\begin{pmatrix} \sin 4t \\ 2\cos 4t \end{pmatrix}$$

And the general solution is $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

Everything appears to be the same except for the extra exponential factor. This makes an enormous difference when it comes to sketching trajectories. This time the trajectories are no longer periodic. The exponential shrinks the ellipse as it is being traced resulting in a spiral. Note that one spiral is sufficient for phase portrait. The only thing that needs to be added is the direction of the shrinking spiral. This can be determined as before by finding the tangent to the trajectory at at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is

$$\begin{pmatrix} -1 & 2 \\ -8 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

This also indicates that the spiral is oriented in the counterclockwise direction. The name of the critical point is easy to remember this time: it is **spiral**. And the critical point is asymtotically stable because all trajectories approach the origin.

The final situation is exemplified by the system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -8 & 1 \end{pmatrix}\mathbf{x}$$

It does not require much imagination to look at the previous example and write down the general solution by analogy:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \quad \text{where} \quad \mathbf{x}_1 = \text{Re}\,\mathbf{x}_d = e^{t}\begin{pmatrix} \cos 4t \\ -2\sin 4t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \text{Im}\,\mathbf{x}_d = e^{t}\begin{pmatrix} \sin 4t \\ 2\cos 4t \end{pmatrix}$$

In this case the critical point is also called spiral. However, this time it is unstable.