

We return to 2×2 autonomous linear homogeneous systems $\mathbf{x}' = A\mathbf{x}$. At the risk of belaboring the obvious, we mention that $\mathbf{0}$ is a critical point of the system, that is $A\mathbf{0} = \mathbf{0}$. Another way of stating this is that the constant solution $\mathbf{x} = \mathbf{0}$ is an equilibrium solution of this system.

Today we will try to sketch phase portraits for 2×2 linear systems. We will see there there are six different types of phase portraits. These are associated with names that are given to their critical points at $\mathbf{0}$.

Before we start let us state two general principles that make this task feasible. First, for the autonomous linear systems, there is a uniqueness and existence guarantee which can be extrapolated from the textbook's Theorem 7.1.2 at the bottom of page 359. A consequence of the uniqueness part is that two trajectories of a linear system of the sort we are considering do not intersect. You have already seen some trajectories of some systems heading towards the origin. However, according to what we have just said, these never reach the origin, because the origin itself is a solution, an equilibrium solution.

The second principle is that nearby trajectories of our systems have nearly the same directions. This is helpful in every trajectory base on just a few trajectories.

To see how the process of sketching a phase portrait works, consider a system $\mathbf{x}' = A\mathbf{x}$ for which we have already found eigenvalues $r_1 = 2$ and $r_2 = -1$ and the corresponding eigenvectors $\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\xi_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. That is, the general solution is

$$\mathbf{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

. Last time we already sketched the basic four trajectories for this system. Now let us consider an arbitrary point in the plane $\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ and graph the trajectory that goes through that point at $t = 0$. Since the first term in \mathbf{x} is dominant for large t , this trajectory must be going away from the origin in a direction parallel to the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for large t . And as t increase from very negative values of t towards zero, the trajectory must be heading towards the origin in a direction parallel to the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The fact this fifth trajectory does not intersect any of the basic four and goes through the point $\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ determines the trajectory rather completely. [Click here for the Direction field Java applet to check your sketch](#)

The critical point with above phase portrait is a called **saddle**. We also introduce some descriptive language concerning the long time behavior of solutions of systems of ODE's. We say that a critical point (bzero in our example) is unstable if there is a trajectory which starts close to the the critical point, yet goes far away as $t \rightarrow \infty$. Note that whenever we have two eigenvalues of opposite sign, 2 of the basic four will approach the origin whereas two of the basic four will go away from the origin with increasing t . Therefore, this will always be unstable and produce a phase portrait with the appearance indicated above.

We now suppose that that the linear system is somewhat different, perhaps is $\mathbf{x}' = B\mathbf{x}$ and that the general solution is found to be:

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

. We note that every solution approaches the critical point at the origin as $t \rightarrow \infty$, including the basic four trajectories. However, only the basic four trajectories maintain their directions of for all values of t in the interval $(-\infty, \infty)$. This is because the term containing e^{-t} is dominant when t is very large (and positive), whereas e^{-2t} is dominant when

t is very negative. Therefore, every trajectory is parallel to the direction of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ when t is very negative and then switch direction to be parallel to $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as t becomes very large and as x approaches the origin. [Click here for the Direction field Java applet to check your sketch](#)

In this case the critical point at the origin is called a **node**. It is also said to be **asymptotically stable** because every solution that starts sufficiently close to the origin eventually approaches the origin. (Note that the words sufficiently close are superfluous in this situation because every trajectory no matter where it starts approaches the origin.) This behavior will occur whenever we look at a linear system with two negative eigenvalues.

Finally, we mention other distinctions between a saddle and a node. An obvious one that comes to mind is that a saddle is always unstable whereas a node can be either asymptotically stable or unstable. A somewhat more sophisticated distinction is that for a saddle every trajectory, except the basic four, runs off to the “frontier” of the xy – *plane* as the variable t approaches infinity and as the variable t approaches minus infinity. In the case of a node this is true for every trajectory but only as the variable t approaches one of infinity and minus infinity but not both (as t approaches the other one, the trajectory must approach the critical point at the origin). (Always remember the trajectories never reach the origin.) This is simply a way of stating geometrically that the eigenvalues have opposite sign.