Math 251 Sections 1,2,10 Lect Notes 10/23/09

We return to a linear 2×2 homogeneous system of first order ODE's with an IVP:

$$\mathbf{x}' = A\mathbf{x} \qquad \mathbf{x}(t_0) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

We search for a solution. In view of the close connection with 2nd order linear constant coefficient ODE's we might expect to find a solution $\mathbf{x}$ of the the form $e^{rt}\xi = e^{rt}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$. Of course we only are interested in finding $e^{rt}\xi$ with ξ not the zero vector because it is rather obvious that $\mathbf{0}$ solves this this system.
Indeed a vector solution of a system represents the solution and the derivative of the solution of the 2nd order ODE.
We plug $\mathbf{x} = e^{rt}\xi$ into the system to see what properties $e^{rt}\xi$ needs to have. For the left hand side we get $\mathbf{x}' = re^{rt}\xi$ whereas for the right hand side we get $A\mathbf{x} = e^{rt}A\xi$. Subtracting these gives

$$\mathbf{0} = A\mathbf{x} - \mathbf{x}' = e^{rt}A\xi - re^{rt}I\xi = e^{rt}\left(A - rI\right)\xi$$

The symbol I denotes the **identity** matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which has the property $I\xi = \xi$ for any ξ. Since e^{rt} is never 0 we must have

$$\left(A - rI\right)\xi = \mathbf{0}$$

Recall that we settled this algebra question already. A nontrivial solution to this algebraic equation exists if $\det\left(A - rI\right) = 0$ This equation is called the **characteristic equation**. The use of the same name for this polynomial equation as for the polynomial equation occuring in the solution of a second order constant coefficient linear homogeneous ODE is not an accident. If one is converted t other the two polynomial equations are the same and therefore should bear the same name.
We start by assuming that that that characteristic equation has two real roots r_1 and r_2. In this case we can find two nonzero vectors ξ_1 ξ_2 with the property

$$\left(A - r_1 I\right)\xi_1 = \mathbf{0} \qquad \textbf{and} \qquad \left(A - r_1 I\right)\xi_1 = \mathbf{0}$$

The numbers r_1, r_2 are called **eigenvalues** and the corresponding vectors ξ_1 and ξ_2 are called **eigenvectors**. Of course one must not confuse which eigenvector belongs to which eigenvalue.
We illustrate these ideas by solving the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\mathbf{x}$$

The characteristic polynomial is:

$$\det\left(A - rI\right) = \det\begin{pmatrix} 1-r & 2 \\ 2 & 1-r \end{pmatrix} = r^{-}2r - 3$$

and this has roots $r_1 = 2$, $r_2 = -1$. We seek the corresponding eigenvectors. For r_1:

$$A - (2)I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Note that at this point you must have a matrix whose rows are multiples of each other. If not there is an error in the calculations already at this point and there is no point in going any further. An eigenvector must be found in less than 8 seconds:

$$\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Of course any nonzero multiple of an eigenvector is an eigenvector. For r_2:

$$A - (-1)I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

Note that again the rows are multiples of each other. In much less than than 8 seconds we find

$$\xi_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So we now have two solutions to the system:

$$\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{x}_2 = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Again, from our experience with the superposition principle we would expect that another solution is

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{for} \quad \text{any} \qquad c_1, c_2$$

Therefore, if we are asked to solve the IVP $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 13 \end{pmatrix}$ then we seek to solve the following pair of linear equations for c_1, c_2:

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{or} \quad \text{equivalently} \qquad \begin{array}{rcl} c_1 - c_2 & = & -1 \\ c_1 + c_2 & = & 13 \end{array}$$

There are four IVP that you should be able to solve instantly.

$$\text{i.} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{ii.} \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad \text{iii.} \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{iv.} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The solutions are

$$\text{i.} \quad e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{ii.} \quad e^{2t} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad \text{iii.} \quad e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{iv.} \quad e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The trajectories of these solutions are also rather important, as you will see next time. So practice sketching them.