Before migrating to systems of ODE's we need to review a well known fact from basic algebra:

The linear homogeneous system of 2 algebraic equations in two unknowns

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

has nontrivial solution (not both x and y are equal to 0) if and only if the determinant of the matrix of coefficients is 0, ie,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \qquad \text{or} \quad \text{equivalently} \qquad ad - bc = 0$$

For example consider the system

$$\begin{aligned} 2x + 4y &= 0 \\ 3x + 6y &= 0 \end{aligned}$$

The determinant of the coefficients is $(2)(6) - (4)(3) = 0$. Therefore nontrivial solutions exist for this system. One possibility is $x = 4$ and $y = -2$. Others are: $x = -4$ and $y = 2$ and $x = 2$ and $y = -1$.

For the next few days we will use vector notation for the 2×2 systems:

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Note that the symbols x and $\mathbf{x}$ denote entirely different objects: $\mathbf{x}$ is a vector and x is its first component and y is its second compontent. In this notation the 2×2 system we started with above is written as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

where $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

In the case where we do have nontrivial solutions it is important to be able to write down one solution very rapidly without thinking a lot. One recipe for doing this is $\begin{pmatrix} b \\ -a \end{pmatrix}$ if not both a and b are zero, otherwise $\begin{pmatrix} d \\ -c \end{pmatrix}$.

We now return to ODE's. We will consider systems of 2 first order ODE's with two unknown functions x and y of the variable t. We have already briefly encountered some very simple systems of ODE's; for example in the falling bodies w/ air resistance problems to find the displacement y we solved a first order ODE for the velocity v and then found the solution of $y' = v$. So we solved a 2×2 systme of first order ODE's.

We will study systems of first order ODE's involving unknown functions x and y of the independent variable t, usually thought of as representing time. They will be written in the following generic form:

$$\begin{aligned} x' &= f(t, x, y) \qquad x(t_0) = \beta \\ y' &= g(t, x, y) \qquad y(t_0) = \alpha \end{aligned}$$

Every second order ODE and IVP can be rewritten as a first order system but not vice versa. Consider for example $y'' + 2y' + 3y = t^4, y(0) = 5, y'(0) = 6$. The following system is equivalent:

$$\begin{aligned} x' &= -2x - 3y + t^4 \qquad x(0) = 6 \\ y' &= x \qquad y(0) = 5 \end{aligned}$$

The functions f and g appearing on the right hand side express formulas involving the inedependent variable t and the unknown function $x = x(t)$ and $y = y(t)$. In addition initial conditions on each of the unknown functions may be specified. Since x and y are both functions of t both right hand sides are ultimately a functions of t. However, it may happen that t does not appear explicity in the formulas. In that case we say the system is **autonomous** (which agrees with the terminology used in the single ODE case). We will only be concerned with autonomous systems of ODE's in this course:

$$\begin{aligned} x' &= f(x,y) \qquad x(t_0)=\beta \\ y' &= g(x,y) \qquad y(t_0)=\alpha \end{aligned}$$

Since the solutions to systems are a pair of functions, it is natural to view them as vector functions: $\begin{pmatrix} x \\ y \end{pmatrix}$. A vector function trace points in the xy-plane as the independent variable t traverses its domain positively. This collection of points together with the direction in which they were traced are called **trajectories**. Trajectories were studied in Math 141 extensively; perhaps they were called parametric curves there. One fact that was established there is that the tangent vector pointing in the positive direction is given by the vector function consisting the the derivatives of x and y: ie, $\begin{pmatrix} x' \\ y' \end{pmatrix}$. We do not need explicit formulas for $\begin{pmatrix} x \\ y \end{pmatrix}$ in order to find $\begin{pmatrix} x' \\ y' \end{pmatrix}$ when $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution to a system of ODE's. The system of ODE's gives this when a specific point is plugged into to the right side of the system. The ease with which we can find tangent vector leads us to the concept of a **Direction Field**. This consists of a bunch of vectors drawn in the xy-plane pointing in the direction of the tangent vector, each one vector being plotted with tail at the point of tangency. The purpose of doing this is to reveal information about the behavior of solutions to the system without solving the system or even having the ability to solve the system explicitly. This can be a tedious process and we are fortunate to have access to computer programs to assist us. You may wish to familiarize yourselves with Professor Mansfield's Direction Field plotter.

Another geometric tool for visualizing the behavior of solutions of systems of ODE's is the **phase portrait**. This is simply a sketch of a few well chosen trajectories from which the sketch of any trajectory for the system can be deduced. These can be produced by clicking on the mouse on the direction field produced by the above mentioned applet. We will be spending a signifcant amount of time during the coming two weeks finding ways of doing the sketching of phase portraits by hand.

A system of ODE's is called **linear** if the functions f and g are linear functions in the x and y. We shall mainly focus on linear systems; but will also attempt to say something about nonlinear ones. Specifically the Predator-Prey system of ODE's and the system describing a damped pendulum.

A linear system can be written as

$$\begin{aligned} x' &= ax+by+h \qquad x(t_0)=\beta \\ y' &= cx+dy+k \qquad y(t_0)=\alpha \end{aligned}$$

We will rewrite this system using matrix notation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix} \qquad \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

and this can be further abbreviated with the notations

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \mathbf{x}(t_0) = \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{b} = \begin{pmatrix} h \\ k \end{pmatrix}$$

The economy in using these notations are significant:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b} \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

We also use the term **homogeneous** to specify the situation $\mathbf{b} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

A critical point for an autonomous system is a value of $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ which make the right hand side equal to $\mathbf{0}$. The significance of this concept lies in the fact that critical points yield equilibrium solutions, ie, solutions to the system consisting of a pair of constant functions: $x = x_0, y = y_0$. Thus, if we are looking at a homogeneous linear system, then the origin is a critical point.

To become familiar with this notation let us find a vector tangent to the solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{b} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ at the point $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Our answer is: $\begin{pmatrix} 1(2) - 2 + 5 \\ 3(2) - 4 + 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$

Also,consider the linear system of 2 ODE's:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$$

We wish to verify that

$$\mathbf{x} = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

solves this 2×2 system. We find that

$$\mathbf{x}' = -e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \textbf{and} \qquad A\mathbf{x} = e^{-t} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-t} \begin{pmatrix} 1 - 2 \\ 2 - 1 \end{pmatrix} = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$