Today we will just go over material connected with Laplace transforms concerning which you have questions.

Also a couple of illustrations of the uses of Laplace transforms:

Suppose an object of mass 1 kgram is suspended from a spring with spring constant equal to 4. At time $t = 0$ the object is at rest at its equilibrium position. At time $t = 1$ an external force of 1 Newton is added to the object. At what time $t = c$ should the external force be removed from the object so that it does it remains at rest at its equilibrium position.

The ODE and IVP for the displacement of this object is:

$$y'' + 4y = u(t-1) - u(t-c) \qquad y(0) = 0 \quad y'(0) = 0$$

We use Laplace transforms to solve. Set $\mathcal{L}\{\mathrm{y}\} = \mathrm{Y}$.

$$s^2Y + 4Y = e^{-s}\frac{1}{s} - e^{-cs}\frac{1}{s}$$

Therefore,

$$Y = e^{-s}\frac{1}{s(s^2+2^2)} - e^{-cs}\frac{1}{s(s^2+2^2)}$$

To find the inverse Laplace we need partial fractions:

$$\frac{1}{s(s^2+2^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2^2}$$

$$1 = A(s^2+2^2) + (Bs+C)s$$

Setting $s = 0$ gives $A = \frac{1}{4}$. Equating corresponding powers of s^2 gives $A + B = 0$, ie, $B = -\frac{1}{4}$ and equating corresponding powers of s gives $C = 0$. So we find that

$$\mathcal{L}\{\mathrm{u(t)}\frac{1}{4}\left(1 - \cos(2\mathrm{t})\right)\} = \frac{1}{\mathrm{s(s^2+2^2)}}$$

And,

$$\mathcal{L}\{\mathrm{u(t-1)}\frac{1}{4}\left(1-\cos(2(\mathrm{t}-1))\right) - \mathrm{u(t-c)}\frac{1}{4}\left(1-\cos(2(\mathrm{t-c}))\right)\} = \mathrm{Y}$$

We now look at $y(t)$ for $0 < t < 1$ and we see that it is 0 as expected. For $1 < t < c$ and we see that it is $\frac{1}{4}\left(1 - \cos(2(t-1))\right)$. This represents oscillations with period π with range $0 \le y \le \frac{1}{2}$. Now we wish to choose c so that

$$\frac{1}{4}\left(1-\cos(2(t-1))\right) - \frac{1}{4}\left(1-\cos(2(t-c))\right) = 0 \quad \text{when} \quad T > c$$

That is, $\cos(2(t-1)) = \cos(2(t-c))$ for $t > c$. This can happen when $2(t-1) = 2(t-c) +$ period of cosine. For example if $2(t-1) = 2(t-c) + 2\pi$, or $c = 1 + \pi$.

Now, suppose an object of mass 1 kgram is suspended from a spring with spring constant equal to 4. At time $t = 0$ the object is released with no initial velocity 1 unit below its equilibrium postion. You wish to stop the motion of the object with one hammer blow applied in the positive direction (ie, downward) at time $t = c$ to the object (which is q times as strong as the unit hammer blow delivered by the unit impulse $\delta(t-c)$.

The ODE and IVP for the displacement of this object is:

$$y'' + 4y = q\delta(t-c) \qquad y(0) = 1 \quad y'(0) = 0$$

We use Laplace transforms to solve. Set $\mathcal{L}\{\mathrm{y}\} = \mathrm{Y}$.

$$s(sY - 1) + 4Y = qe^{-cs}$$

Therefore,

$$Y = \frac{s}{s^2+2^2} + qe^{-cs}\frac{1}{s^2+2^2}$$

The inverse Laplace transform is:

$$\mathcal{L}\{\cos 2\mathrm{t} + \frac{\mathrm{q}}{2}\mathrm{u}(\mathrm{t}-\mathrm{c})\sin 2(\mathrm{t}-\mathrm{c})\} = \mathrm{Y}$$

We see that initially $y(t)$ is simply a cosine function. Then as t passes c, the displacement $y(t)$ becomes

$$y(t) = \cos 2t + \frac{q}{2}\sin 2(t-c)$$

and we wish to choose c and q so that this is zero. Well, the correct choice for q is obvious; one cannot expect cancelation to occur unless the two trig functions involved have the same amplitude. This means that $q/2 = 1$ or $q = 2$. Now in order to choose c we observe that sine shifted $\frac{\pi}{2}$ units to the right is the negative of cosine. Therefore, we need choose c so that the variable of sine, $2(t-c)$ is the variable of cosine $2t$ minus $\frac{\pi}{2}$ plus a period:

$$2t - \frac{\pi}{2} = 2(t-c) + \text{ period of sine}$$

For example

$$2t = 2(t-c) + \frac{\pi}{2} +$$

That is

$$c = \frac{\pi}{4} + \frac{1}{2}\text{ period of sine}$$

For example $c = \frac{\pi}{4} \approx 0.7854$ or $c = \frac{5\pi}{4} \approx 3.927$

If you do not have a lot of confidence in math, then perhaps seeing the following screen capture of the Java applet will build some: