Today we introduce a mathematical model for collisions such as the one that occurs when a hammer strikes a steel object. The problem people have with understanding this model is that it stipulates that this collision occurs at exactly one moment in time. However, the actual visualization of collisions that people have is that it occurs over a very short interval of time.

We define the Dirac Delta $\delta(t)$, aka unit impulse, as follows: δ is a force which when acting on an object, increases the momentum of the object by 1 unit at time $t = 0$ but does absolutely nothing at any time other at $t = 0$.

We immediately look at an example to clarify the meaning of this definition. Suppose that the acceleration of an object of mass 1 kgram is given by $y''(t) = \delta(t-2)$. If $y(0) = 1$ and $y'(0) = 0$, then we sketch a graph for $y'(t)$ and $y(t)$ and from also find a formula for $y'(t)$ and $y(t)$.

For the graph of $y'(t)$ we use Newton's first law of motion which states that "Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed". Since $\delta(t-2)$ is the "impressed" force and it does nothing except at $t = 2$, the graph of the velocity our object is a horizontal line at all points except at $t = 2$. In particular, to the left of $t = 2$ it is a horizontal line passing through the origin. At $t = 2$ the velocity takes a jump of one unit, since by definition $\delta(t-2)$ increases the momentum by one unit (momentum and velocity are the same for an object of mass 1kg.) therefore to the right of $t = 2$ the graph of y' is a horizontal line situated one unit above the t-axis. This in effect says that $y' = u(t-2)$, the unit step function.

Having the graph and formula for y', we can now easily figure out the graph of $y(t)$. Since y' is zero to the left of $t = 2$ the graph of $y(t)$ is a horizontal line there passing through the point $(0, y(0)) = (0, 1)$. Since y' changes to 1 as t passes $t = 2$, the slope of $y(t)$ changes to 1 and remains there for all $t > 2$.

It is now a simple matter to derive the Laplace transform of $\delta(t-2)$. We use the derivative formula which states that

$$\mathcal{L}\{\mathrm{y}''\} = \mathrm{s}\mathcal{L}\{\mathrm{y}'\} - \mathrm{y}(0)$$

In our example $y"(t) = \delta(t-2)$ and $y' = u(t-2)$ and also $y'(0) = 0$. Therefore

$$\mathcal{L}\{\delta(\mathrm{t}-2)\} = \mathrm{s}\mathcal{L}\{\mathrm{u}(\mathrm{t}-2)\} - 0 = \mathrm{s}\left(\mathrm{e}^{-2\mathrm{s}}\frac{1}{\mathrm{s}}\right) = \mathrm{e}^{-2\mathrm{s}}$$

It is a simple matter to verify that in general:

$$\mathcal{L}\{\delta(\mathrm{t}-\mathrm{c})\} = \mathrm{e}^{-\mathrm{cs}}$$

This is the last of the eight formulas that are required for the examination covering Laplace transforms. Observe that when solving differential equations this Laplace transform formula will never be needed to be invoked in the process of finding the inverse Laplace transform.

Solve the following IVP: $y" - y = \delta(t-2) \quad y(0) = 1, y'(0) = -1$

We take the Laplace transform of both sides:

$$s(sY-1) + 1 - Y = e^{-2s}$$

Solving for Y gives:

$$Y = \frac{s-1}{s^2-1} + e^{-2s}\frac{1}{s^2-1} = \frac{1}{s+1} + e^{-2s}\frac{1}{s^2-1}$$

We will need the following partial fraction expansions:

$$\frac{1}{s^2-1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

Ignoring the e^{-2s} we find

$$\mathcal{L}\{\frac{1}{2}\left(\mathrm{e}^{\mathrm{t}}-\mathrm{e}^{-\mathrm{t}}\right)\}=\frac{1}{\mathrm{s}^2-1}$$

Therefore by the shift formula

$$\mathcal{L}\{\mathrm{u}(\mathrm{t}-2)\frac{1}{2}\left(\mathrm{e}^{\mathrm{t}-2}-\mathrm{e}^{-(\mathrm{t}-2)}\right)\}=\mathrm{e}^{-2\mathrm{s}}\frac{1}{\mathrm{s}^2-1}$$

Finally,

$$y(t)=e^{-t}+\frac{1}{2}u(t-2)\left(e^{t-2}-e^{-(t-2)}\right)$$

We have already seen that the velocity of an object that undergoes a collision has a jump discontinuity. One may think that a large collision could cause a discontinuity in the position of the object. As a matter of fact it is easy to see that $\lim_{t\to 2^-} y(t)=1$ and $\lim_{t\to 2^+} y(t)=1$ which demonstrates that this is not the case.