Math 251 Sections 1,2,10 Lec Notes 10/16/09

Last time we introduced the formula for finding the Laplace transform of functions involving the unit step function:

$$\text{If} \quad \mathcal{L}\{\mathrm{u(t)f(t)}\} = \mathrm{F(s)} \quad \text{then} \quad \mathcal{L}\{\mathrm{u(t-c)f(t-c)}\} = \mathrm{e^{-cs}F(s)}$$

We illustrated how it can be used. We saw that frequently there was a bit of a complication because frequently we need to use it to find the Laplace transform of a product with the shift on variable of u and and the shift on the variable of f not being the same. There is no such difficulty in using the above shift formula to find inverse Laplace transform of $e^{-cs}F(s)$ whenever it is possible to find the inverse Laplace transform of $F(s)$. The process is clear. We list the steps needed to execute it:

I. Ignore e^{-cs}, temporarily.
II. Find $f(t)$ so that $\mathcal{L}\{\mathrm{u(t)f(t)}\} = \mathcal{L}\{\mathrm{f(t)}\} = \mathrm{F(s)}$
III. Make sure $f(t)$ is multiplied by $u(t)$.
IV. Replace every occurence of t in $u(t)f(t)$ by $(t-c)$. (Make sure adequate parenthesis are used.)

We illustrate this by finding the inverse Laplace transform of $e^{-6s}\dfrac{1}{s+7}$. For this purpose we first observer that $\mathcal{L}\{\mathrm{e^{-7t}}\} = \dfrac{1}{\mathrm{s+7}}$, ignoring the $^{-6s}$. We then multiply by $u(t)$ and then shift every t to $t-6$:

$$\mathcal{L}\{\mathrm{u(t-6)e^{7(t-6)}}\} = \mathrm{e^{-6s}}\frac{1}{\mathrm{s-7}}$$

Let's use Laplace transforms to solve the following IVP:

$$y" + 9y = u(t-4) \quad y(0) = 1,\ y'(0) = 0$$

We set $\mathcal{L}\{\mathrm{y(t)}\} = \mathrm{Y(s)}$ and take Laplace transforms of both sides:

$$s^2Y - s + 9Y = e^{-4s}\frac{1}{s}$$

and therefore

$$Y = \frac{s}{s^2+3^2} + e^{-4s}\frac{1}{s(s^2+3^2)}$$

We easily find the partial fraction expansion:

$$\frac{1}{s(s^2+3^2)} = \frac{1}{9}\left(\frac{1}{s} - \frac{s}{s^2+3^2}\right)$$

Therefore,

$$\mathcal{L}\{\frac{1}{9}\,(1-\cos 3\mathrm{t})\} = \frac{1}{\mathrm{s(s^2+3^2)}}$$

and

$$\mathcal{L}\{\frac{1}{9}\,(1-\cos 3(\mathrm{t}-4))\,\mathrm{u(t-4)}\} = \frac{\mathrm{e^{-4s}}}{\mathrm{s(s^2+3^2)}}$$

Finally,

$$\mathcal{L}\{\cos 3\mathrm{t} + \frac{1}{9}\,(1-\cos 3(\mathrm{t}-4))\,\mathrm{u(t-4)}\} = \mathrm{Y}$$

Let's now use Laplace transforms to solve the following IVP:

$$y' + 2y = tu(t-1) \quad y(0) = 1$$

We set $\mathcal{L}\{\mathrm{y(t)}\} = \mathrm{Y(s)}$. We start taking Laplace transforms of both sides:

$$sY - 1 + 2Y = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

Therefore,

$$Y \quad = \quad \frac{1}{s+2} + e^{-s}\left(\frac{1}{s^2(s+2)} + \frac{1}{s(s+2)}\right)$$

We easily find the partial fraction expansion:

$$\frac{1}{s^2(s+2)} + \frac{1}{s(s+2)} = \frac{1/4}{s} + \frac{1/2}{s^2} - \frac{1/4}{s+2}$$

Therefore,

$$\mathcal{L}\{\frac{1}{4} + \frac{1}{2}\mathrm{t}\} - \frac{1}{4}\mathrm{e}^{-2\mathrm{t}} = \left(\frac{1}{\mathrm{s}^2(\mathrm{s}+2)} + \frac{1}{\mathrm{s}(\mathrm{s}+2)}\right)$$

And we finally arrive at:

$$\mathcal{L}\{\mathrm{e}^{-2\mathrm{t}} + \mathrm{u}(\mathrm{t}-1)\left(\frac{1}{4} + \frac{1}{2}(\mathrm{t}-1) - \frac{1}{4}\mathrm{e}^{-2(\mathrm{t}-1)}\right)\} = \mathrm{Y}$$