Most people who finished two or three semesters of calculus are left with the impression that discontinuous functions do not deserve the same attention that continuous functions receive or should be avoided altogether. That is really not the case. Many significant physical phenomena can only be modeled accurately by discontinuous functions. For this reason, in the remainder of our study of Laplace transforms, as well as the last quarter of this course, discontinuous functions will be close to center of our attention.

We consider to basic discontinuous functions: the unit step function, also known as the Heaviside function, and next week the Dirac Delta.

The unit step function, denoted by $u(t)$ is defined for all t in the following two parts:

$$u(t) = \begin{cases} 0, & \text{if} \quad t < 0 \\ 1, & \text{if} \quad 0 \leq t \end{cases}$$

Right now the value of $u(0) = 1$ is not siginificant. It just as well could have been chosen to be $u(0) = 0$. As a matter of fact, when we will discuss Fourier series we will see that the average of the 1-sided limits is a preferred choice for some some purposes.

Having $u(t)$ defined we can build other functions that have jump discontinuities. For example, shifting $u(t)$ to the right c units produces the unit step function with jump discontinuity at $t = c$.

$$u(t-c) = \begin{cases} 0, & \text{if} \quad t < c \\ 1, & \text{if} \quad c \leq t \end{cases}$$

The step-up step-down function $f(t) = \begin{cases} 0, & \text{if} \quad t < 1 \\ 1, & \text{if} \quad 1 \leq t < 2 \\ 0, & \text{if} \quad 2 \leq t \end{cases} = u(t-1) - u(t-2)$

One can also think of using a bunch of u's shifted progressively to the right to describe a staircase:

$$u(t-1) + u(t-2) + u(t-3) + u(t-4) + u(t-5) + \ldots$$

Of course, not all of the stairs need be the same height and some in fact my be steps down and not all need to be the same lenth. For example,

$$f(t) = \begin{cases} 0, & \text{if} \quad t < 1 \\ 1, & \text{if} \quad 3 \leq t < 4 \\ 2, & \text{if} \quad 4 \leq t < 5 \\ -3, & \text{if} \quad 5 \leq t \end{cases} = u(t-3) - u(t-4) + 2(u(t-4) - u(t-5)) - 3u(t-5) = u(t-3) + u(t-4) - 5u(t-5))$$

In this fashion any configuration of steps can be expressed in terms of linear combinations of the unit step function shifted. This includes an infinite sequence of steps such as the square sine wave; we will not cover the Laplace transform of such a series although they do appear in the exercises in your textbook.

Another collection of useful functions are the sawtooth and ramp functions. For example,

$$f(t) = \begin{cases} 0, & \text{if} \quad t < 1 \\ t-1, & \text{if} \quad 1 \leq t < 2 \\ 0, & \text{if} \quad 2 \leq t \end{cases} = (t-1)(u(t-1) - u(t-2))$$

Or,

$$f(t) = \begin{cases} 0, & \text{if} \quad t < 1 \\ t-1, & \text{if} \quad 1 \leq t < 3 \\ 0, & \text{if} \quad 3 \leq t < 4 \\ 2, & \text{if} \quad 4 \leq t \end{cases} = (t-1)(u(t-1) - u(t-3)) + 2u(t-4)$$

Before going to the Laplace transform of formula involving the unit step function, we need to point out that every Laplace transform actually contains a unit step function. Ie, for every function which has a Laplace transform

$$\mathcal{L}\{\mathrm{f(t)}\} = \mathcal{L}\{\mathrm{u(t)f(t)}\}$$

Of course, this is a triviality which may seem not even worthwhile mentioning. But now the formula

$$\text{If} \quad \mathcal{L}\{\mathrm{u(t)f(t)}\} = \mathrm{F(s)} \quad \text{then} \quad \mathcal{L}\{\mathrm{u(t-c)f(t-c)}\} = \mathrm{e^{-cs}F(s)}$$

Can be seen to symmetric with with the first shift formula

$$\text{If} \quad \mathcal{L}\{\mathrm{u(t)f(t)}\} = \mathrm{F(s)} \quad \text{then} \quad \mathcal{L}\{\mathrm{e^{at}u(t)f(t)}\} = \mathrm{F(s-a)}$$

Noting this symmetry does not constitute a proof of the formula. However, the formal proof follows directly from the definition and a simple change of variable:

$$\mathcal{L}\{\mathrm{u(t-c)f(t-c)}\} = \int_0^\infty \mathrm{e^{-st}u(t-c)f(t-c)\,dt} = \int_c^\infty \mathrm{e^{-st}u(t-c)f(t-c)\,dt}$$

because $u(t-c) = 0$ when $0 < t \leq c$ We set $u = t - c$. Then $du = dt$ and $t = u + c$. We obtain

$$\mathcal{L}\{\mathrm{u(t-c)f(t-c)}\} = \int_0^\infty \mathrm{e^{-s(u+c)}f(u)\,du} = \int_0^\infty \mathrm{e^{-su}e^{-cs}f(u)\,du} = \mathrm{e^{-cs}F(s)}$$

Let us now return to compute the Laplace transforms of a couple of the functions above.

For example the Laplace transform of
$f(t) = u(t-3) - u(t-4) + 2(u(t-4) - u(t-5)) - 3u(t-5) = u(t-3) + u(t-4) - 5u(t-5))$
is
$\mathcal{L}\{\mathrm{f(t)}\} = \mathrm{e^{-3s}\frac{1}{s} + e^{-4s}\frac{1}{s} - 5e^{-5s}\frac{1}{s} = \frac{1}{s}\left(e^{-3s} + e^{-4s} - 5e^{-5s}\right)}$ And, for example

$$f(t) = (u(t-1) - u(t-2)) + 2u(t-2) = u(t-1) + u(t-2)$$

has

$$\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$$

And, for example the Laplace transform of $f(t) = (t-1)(u(t-1) - u(t-3)) + 5u(t-4) = (t-1)u(t-1) - (t-3)u(t-3) - 2u(t-1) + 5u(t-4)$
is $\mathcal{L}\{\mathrm{f(t)}\} = \mathrm{e^{-s}\frac{1}{s^2} - e^{-3s}\frac{1}{s^2} - 2e^{-3s}\frac{1}{s} + 5e^{-4s}\frac{1}{s}}$

Finally, what about the Laplace transform of the following function: $sin(t)u(t-\pi/2)$? We cannot apply the above shift formula directly to find $\mathcal{L}\{\mathrm{sin(t)u(t-\pi/2)}\}$. However, we can try to replace $\sin(t)$ by $\sin(t-\pi/2)$ or perhaps even $\cos(t-\pi/2)$ in order to make it a more tractable problem. With this in we complare the graphs of $\sin(t)$ and $\sin(t-\pi/2)$ and $\cos(t-\pi/2)$ and we easily see that $\sin(t) = -\cos(t-\pi/2)$. Therefore,

$$\mathcal{L}\{\mathrm{sin(t)u(t-\pi/2)}\} = -\mathcal{L}\{\mathrm{cos(t-\pi/2)u(t-\pi/2)}\} = -\mathrm{e^{-\pi s/2}}\frac{\mathrm{s}}{\mathrm{s^2+1^2}}$$