Today we introduce the sixth Laplace transform formula. If we would be permitted to switch order of improper integration then it would be a very simple consequence of the defintion of Laplace transform. Indeed, if $F(s)=$ $\mathcal{L}\{\mathrm{f}(\mathrm{t})\}$, then

$$
\begin{aligned}
\frac{d}{d s} F(s) & =\frac{d}{d s} \int_{0}^{\infty} e^{-s t} f(t) d t \\
& =\int_{0}^{\infty} \frac{\partial}{\partial s} e^{-s t} f(t) d t \\
& =\int_{0}^{\infty}-t e^{-s t} f(t) d t \\
& =-\mathcal{L}\{\mathrm{tf}(\mathrm{t})\}
\end{aligned}
$$

We will usually rewrite this formual as:

$$
\mathcal{L}\{\mathrm{tf}(\mathrm{t})\}=-\frac{\mathrm{d}}{\mathrm{ds}} \mathrm{~F}(\mathrm{~s})
$$

This is a very useful formula. For example knowing that that $\mathcal{L}\{1\}=\frac{1}{\mathrm{~s}}$ gives us the Laplace transform of $t$ : $\mathcal{L}\{\mathrm{t}(1)\}=-\frac{\mathrm{d}}{\mathrm{ds}} \frac{1}{\mathrm{~s}}=--\frac{1}{\mathrm{~s}^{2}}=\frac{1}{\mathrm{~s}^{2}}$.
Observe the symmetry between the two differentiatiation formulas that we now have:

$$
\mathcal{L}\{\mathrm{tf}(\mathrm{t})\}=-\frac{\mathrm{d}}{\mathrm{ds}} \mathrm{~F}(\mathrm{~s}) \quad \mathcal{L}\left\{\mathrm{f}^{\prime}(\mathrm{t})\right\}=\mathrm{sF}(\mathrm{~s})-\mathrm{f}(0)
$$

They say basically that differentiation one side of a Laplace transform formula requires multiplication by the independent variable that resides on the other side. Then there are two minor details that need to be added: a minus sign if the derivative is on the right and adjustment by the initial value of the function to which Laplace transform is applied if the differentiation is on the left.
We now solve some IVP's for ODE's which require the use of this new formula.

$$
y^{\prime \prime}-5 y^{\prime}+6 y=e^{2 t} \quad y(0)=-1 \quad y^{\prime}(0)=1
$$

We let $Y=\mathcal{L}\{y\}$. Then taking Laplace transforms of both sides gives:

$$
\begin{gathered}
s(s Y+1)-1-5(s Y+1)+6 Y=\frac{1}{s-2} \\
s^{2} Y+s-1-5 s Y-5+6 Y=\frac{1}{s-2}
\end{gathered}
$$

Solving for $Y$ gives

$$
Y=\frac{-s+6}{(s-2)(s-3)}+\frac{1}{(s-2)^{2}(s-3)}
$$

We use partial fraction to rewrite the right hand side as follows:

$$
\frac{s+6}{(s-2)(s-3)}+\frac{1}{(s-2)^{2}(s-3)}=\frac{-5}{s-2}-\frac{1}{(s-2)^{2}}+\frac{4}{s-3}
$$

Therefore the solution is

$$
y=-5 e^{2 t}-t e^{2 t}+4 e^{3 t}
$$

et us solve the following IVP:

$$
y^{\prime \prime}-2 y^{\prime}+y=e^{t} \quad y(0)=1 \quad y^{\prime}(0)=1
$$

We start by letting $Y=\mathcal{L}\{y\}$. Then taking Laplace transforms of both sides gives:

$$
s(s Y-1)-1-2(s Y-1)+Y=\frac{1}{s-1}
$$

$$
s^{2} Y-s-1-2 s Y+2+Y=\frac{1}{s-1}
$$

Solving for $Y$ gives

$$
Y=\frac{s-1}{(s-1)^{2}}+\frac{1}{(s-1)^{3}}=\frac{1}{s-1}+\frac{1}{(s-1)^{3}}
$$

Therefore the solution is

$$
y=e^{t}-\frac{1}{2} t^{2} e^{t}
$$

Now let's use Laplace transforms to solve the following IVP: $\quad y^{\prime \prime}+25 y=\cos 5 t \quad y(0)=1, \quad y^{\prime}(0)=-1$
Let $\mathcal{L}\{\mathrm{y}(\mathrm{t})\}=\mathrm{Y}(\mathrm{s})$. We start taking Laplace transforms of both sides by observing that $\mathcal{L}\left\{\mathrm{y}^{\prime}(\mathrm{t})\right\}=\mathrm{sY}+1$. We obtain

$$
s(s Y-1)+1+5 Y=\frac{s}{s^{2}+5}
$$

Solving for $Y$ gives

$$
Y=\frac{s-1}{s^{2}+5^{2}}+\frac{s}{\left(s^{2}+5^{2}\right)^{2}}
$$

It is easy to recognize the function which Laplace transforms into the first term on the right side of the above equation:

$$
\mathcal{L}\left\{\cos \mathrm{t} 5-\frac{1}{5} \sin 5 \mathrm{t}\right\}=\frac{\mathrm{s}-1}{\mathrm{~s}^{2}+5^{2}}
$$

For the second term we note that its denominator is the square of the term that normally appears in the denominator of the Laplace transform of sine or cosine. We also recall that differentiating a linear polynomial divided by a quadratic causes the denominator to be squared (remember the quotient rule). These clues lead us to believe that the second term might be the derivative of the Laplace transform of sine or cosine. To see which one we write down both derivatives

$$
\frac{d}{d s} \frac{s}{s^{2}+5^{2}}=\frac{1}{s^{2}+5^{2}}-\frac{2 s^{2}}{\left(s^{2}+5^{2}\right)^{2}} \quad \frac{d}{d s} \frac{5}{s^{2}+5^{2}}=-\frac{10 s}{s^{2}+5^{2}}
$$

And now we clearly see that $\frac{1}{10} \mathcal{L}\{\mathrm{t} \sin 5 \mathrm{t}\}$ is equal to the second term above. Therefore,

$$
y=\cos 5 t-\frac{1}{5} \sin 5 t+\frac{1}{10} \mathcal{L}\{\mathrm{t} \sin 5 \mathrm{t}\}
$$

is the solution we are seeking.
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