We begin where we left off last time: we use Laplace transforms to solve the following IVP for a second order linear ODE: $y'' + 5y' + 4y = 0 \quad y(0) = 1, \quad y'(0) = 2$
We let $\mathcal{L}\{y\} = Y$ and start taking Laplace transforms of both sides working from right to left. We get $\mathcal{L}\{4y\} = 4Y$ and $\mathcal{L}\{5y'\} = 5(sY-1)$. But, it looks like we are missing a formula for $\mathcal{L}\{y''\}$. Actually, we are not missing anything because $\mathcal{L}\{y''\}$ is related to $\mathcal{L}\{y'\}$ in the same way as $\mathcal{L}\{y'\}$. That is, $\mathcal{L}\{y''\} = s\mathcal{L}\{y'\} - y'(0) = s(sY-1) - 2$. Putting these three terms together gives us the Laplace transform of the left hand side:

$$s(sY-1) - 2 + 5(sY-1) + 4Y = 0$$

$$(s^2 + 5s + 4)Y = s + 7$$

Now solving for Y

$$\begin{aligned} Y &= \frac{s+7}{s^2+5s+4} = \frac{s+7}{(s+1)(s+4)} \\ &= \frac{2}{s+1} - \frac{1}{s+4} \\ &= \mathcal{L}\{2e^{-t} - e^{-4t}\} \end{aligned}$$

That is,

$$y(t) = 2e^{-t} - e^{-4t}$$

Today we will develop a new formula that is needed to solve some simple second order ODE's using Laplace transforms. To motivate this formula we recall two formulas from last time:

$$\begin{aligned} \mathcal{L}\{1\} &= \frac{1}{s} \\ \mathcal{L}\{e^{at}\} &= \frac{1}{s-a} \end{aligned}$$

Comparing these two formulas we observe that the function on the left hand sides which we are taking Laplace transform of are related by the fact that the second is e^{at} and the functions on the right are related by the fact that the variable in the first is replace by the variable shifted a units to the right in the second.

We ask ourselves whether or not this is a general fact. Specifically, is the following true:

$$\begin{aligned} \text{If} \quad \mathcal{L}\{f(t)\} &= F(s) \\ \text{then} \quad \mathcal{L}\{e^{at}f(t)\} &= F(s-a)\ ? \end{aligned}$$

We check this by replacing $f(t)$ in the definition of Laplace transform by $e^{at}f(t)$:

$$\begin{aligned} \mathcal{L}\{e^{at}f(t)\} &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,dt \end{aligned}$$

We compare the right hand side above with the defintion of Laplace transform $F(s) = \int_0^\infty e^{-st}f(t)\,dt$ and we see that the s has been replaced by $s-a$. Therefore, it in fact is $F(s-a)$ which is what we expected.

We apply this to finding Laplace transforms of the following: $e^{-t}\sin 2t$ and $e^{-t}\cos 2t$. Indeed, since $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+2^2}$, we see that $\mathcal{L}\{e^{-t}\sin 2t\} = \frac{2}{(s+1)^2+2^2}$

since $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+2^2}$ we see that $\mathcal{L}\{e^{-t}\cos 2t\} = \frac{(s+1)}{(s+1)^2+2^2}$ Observe that in the last formula both s's are shifted from s to $s+1$.

We now reverse the question: Find the function whose Laplace transform is: $\frac{1}{s^2+14s+74}$. We begin by completing the square in the denominator:

$$\begin{aligned} \frac{1}{s^2+14s+74} &= \frac{1}{(s+7)^2+5^2} \\ &= \frac{1}{5}\mathcal{L}\{\mathrm{e}^{-7\mathrm{t}}\sin 5\mathrm{t}\} \\ \text{because}\quad \frac{1}{5}\mathcal{L}\{\sin 5\mathrm{t}\} &= \frac{1}{s^2+5^2} \end{aligned}$$

We try the same approach to finding the inverse Laplace transform of a similiar function:

$$\frac{s}{s^2+14s+74} = \frac{s}{(s+7)^2+5^2}$$

This time it is not possible to recognize a shift of the variable s from s to $s+7$ because the variable appears once shifted and once not shifted. A very simple algebraic trick can help us here:

$$\begin{aligned} \frac{s}{s^2+14s+74} &= \frac{s+7}{(s+7)^2+5^2} - \frac{7}{(s+7)^2+5^2} \\ &= \mathcal{L}\{\mathrm{e}^{-7\mathrm{t}}\cos 5\mathrm{t} - \frac{7}{5}\mathrm{e}^{-7\mathrm{t}}\sin 5\mathrm{t}\} \\ \text{because}\quad \mathcal{L}\{\cos 5\mathrm{t}\} &= \frac{s}{s^2+5^2} \\ \text{and}\quad \frac{1}{5}\mathcal{L}\{\sin 5\mathrm{t}\} &= \frac{1}{s^2+5^2} \end{aligned}$$

With these preliminaries out of the way we are ready to solve 2nd order constant coefficient linear ODE's: $y'' + 14y' + 74y = 0 \quad y(0) = 1, \quad y'(0) = 2$ The first step is to take Laplace transforms of both sides. It is also convenient to work from right to left in doing this. The Laplace transform of the right hand side is easy: 0. The Laplace transform of the left is

$$s^2Y - s - 2 + 14(sY - 1) + 74Y = 0$$

$$(s^2 + 14s + 50)Y = s + 16$$

(Note that the characteristic polynomial always makes an appearance even when using Laplace transforms to solve the linear constant coefficient ODE). Now solving for Y

$$\begin{aligned} Y &= \frac{s+16}{s^2+14s+74} \\ &= \frac{s+7}{(s+7)^2+25} + 9\frac{1}{(s+7)^2+25} \\ &= \mathcal{L}\{\mathrm{e}^{-7\mathrm{t}}(\cos\mathrm{t} + \frac{9}{5}\sin\mathrm{t})\}\} \end{aligned}$$

That is,

$$y(t) = e^{-7t}(\cos t + \frac{9}{5}\sin t)$$