

As preparation to Laplace transforms we recall a couple of ideas and a formula usually covered in Math 141.

A function f is said to be **piecewise continuous** on a closed interval if it has only a finite number of discontinuities, which are no worse than jump discontinuities. Ie, the one sided limits exist but the reason for the discontinuity is that they do not agree. It is piecewise continuous **everywhere** if it is piecewise continuous on every closed interval. We also note that many of the properties of integral studied in Math 141 carry over to piecewise continuous functions. Indeed the integral over an interval is simply the sum of the integrals over the finite number of subintervals on which the function is continuous.

Specifically, we need to recall the definition of improper integral:

$$\int_0^{\infty} f(t) dt = \lim_{A \rightarrow \infty} \int_0^A f(t) dt$$

As an example let us compute the following: $\int_0^{\infty} e^{ct} dt$ In case $c = 0$ it is obvious that this improper integral diverges. We assume $c \neq 0$ and we plug into the above definition:

$$\begin{aligned} \int_0^{\infty} e^{ct} dt &= \lim_{A \rightarrow \infty} \int_0^A e^{ct} dt = \frac{1}{c} \lim_{A \rightarrow \infty} [e^{ct}]_0^A \\ &= \frac{1}{c} \lim_{A \rightarrow \infty} (1 - e^{cA}) \end{aligned}$$

We observe that the last limit does not exist if $c > 0$ and it is equal to $\frac{-1}{c}$ whenever $c < 0$.

The Laplace transform is a device that enables us to replace the process of solving IVP's for linear constant coefficient differential equations with solving linear algebraic equations.

Although the table of Laplace transforms given in the textbook will be provided for the examination, it is a losing battle to try to use that table to solve problems. A much better strategy is to memorize the eight Laplace transform given out with today's FAQ, instead of trying to navigate through the maze of 19 formulas which appear in the textbook's table.

The definition of the Laplace transforms of $f(t)$, a piecewise continuous function defined on $[0, \infty)$ is as follows. If we set $\mathcal{L}\{f(t)\} = F(s)$, then

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

A general property of Laplace transform is that it satisfies the Superposition Principle, ie, the Laplace transform of a linear combination of two function can be found by taking the same linear combination of the Laplace transforms.

Another property that $f(t)$ must have in order for the improper integral, and hence the Laplace transform to converge is that it have exponential growth. This means that we can find constants M and m such that $|f(t)| \leq Me^{at}$ for all sufficiently large t . Obviously, e^{at} has this property. Also any bounded function has this property. But e^{t^2} does not have this property. Because if $e^{t^2} \leq Me^{mt}$ for some M and m and all sufficiently large t , then taking natural logarithms of both sides would imply that $t^2 \leq \ln M + mt$ for all sufficiently large. But it is impossible for a parabola to stay below a fixed line for all sufficiently large t .

Our first example of calculating the Laplace transform is $\mathcal{L}\{e^{at}\}$, for some constant a . We apply the definition:

$$\begin{aligned} \mathcal{L}\{e^{at}\} &= \int_0^{\infty} e^{-st} e^{at} dt \\ &= \int_0^{\infty} e^{-(s-a)t} dt \\ &= \frac{-1}{-(s-a)} = \frac{1}{s-a} \end{aligned}$$

Here we used the formula for the improper integral of e^{ct} which we derived last time. Of course, the Laplace transform only exists when $s - a < 0$, ie, $s > a$. An important special case of this $a = 0$ which gives the formula: $\mathcal{L}\{1\} = \frac{1}{s}$. Although the derivation of the above depends on a being real, if we believe the above formula also for complex values of a , then we easily arrive at the formulas for the Laplace transform of sine and cosine.

That is, we set $a = ib$ and use

$$\mathcal{L}\{e^{ibt}\} = \frac{1}{s - ib} = \frac{s + ib}{s^2 + b^2}$$

Separating real and imaginary parts of the above gives the formulas we seek:

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2} \quad \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2 + b^2}$$

Our goal for today is to solve a simple first order linear ODE with initial value problem. For this purpose we need one more formula which expresses the Laplace transform of the derivative of a function in terms the Laplace transform of the original function.

We have a clue to this in the above formulas for the Laplace transform of sine and cosine. To make the clue more obvious we rewrite the above with the constant b replaced by 1:

$$\mathcal{L}\{\sin(t)\} = \frac{1}{s^2 + 1} \quad \mathcal{L}\{\cos(t)\} = \frac{s}{s^2 + 1}$$

Since cosine is the derivative of sine and since the right hand side of the second formula in the line above is s times the right hands side of the first formula, one might conjecture the following

$$\text{If } \mathcal{L}\{g(t)\} = G(s), \text{ then } \mathcal{L}\{g'(t)\} = sG(s)$$

However, thinking a moment it is obvious this cannot be true in general because the derivative of cosine is negative sine, nevertheless $-s$ times the right hand side in the second formula does not give the right had side in the first.

On the other hand, perhaps flaw in our conjecture is due to the fact that sine is equal to zero at $t = 0$ and cosine is not. So let us amend our conjecture as follows:

$$\text{If } g(0) = 0 \text{ and if } \mathcal{L}\{g(t)\} = G(s), \text{ then } \mathcal{L}\{g'(t)\} = sG(s)$$

It turns out that this is correct. If we are given a general piecewise continuous $f(t)$ then we set $g(t) = f(t) - f(0)$ and apply our modified conjecture to $g(t)$ which does have $g(0) = 0$ to conclude that

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= \mathcal{L}\{g'(t)\} = sG(s) \\ &= s(\mathcal{L}\{f(t)\} - \mathcal{L}\{f(0)\}) = s(F(s) - \mathcal{L}\{f(0)\}) \\ &= s\left(F(s) - f(0)\frac{1}{s}\right) = sF(s) - f(0) \end{aligned}$$

Having this formula at our disposable, enables us to solve the following IVP using Laplace transforms:

$$y' + 2y = e^{2t}, \quad y(0) = 3$$

Indeed set $\mathcal{L}\{y(t)\} = Y(s)$, new unknown function of s . Then taking Laplace transforms of each side gives the following algebraic equation involving Y :

$$sY - 3 + 2Y = \frac{1}{s - 2}$$

Solving for Y gives:

$$Y = \frac{3}{s + 2} + \frac{1}{(s - 2)(s + 2)}$$

It is easy to see that the first term on the right hand side is: $\mathcal{L}\{3e^{-2t}\}$. What about the second term? We recall that by partial fractions:

$$\frac{1}{(s - 2)(s + 2)} = \frac{1/4}{s - 2} - \frac{1/4}{s + 2}$$

Therefore, $\frac{1}{4}\mathcal{L}\{e^{2t}\} - \frac{1}{4}\mathcal{L}\{e^{-2t}\}$ is the second term on the right hand side. We conclude that

$$Y = \frac{11}{4}\mathcal{L}\{e^{2t}\} - \frac{1}{4}\mathcal{L}\{e^{-2t}\}$$

and finally arrive at the solution

$$y = \frac{11}{4}e^{2t} - \frac{1}{4}e^{-2t}$$