Today we consider the effect of a periodic external force on a spring-mass system either with or without a damping device. The external force $F(t)$ is assumed to be either $\sin(\omega t)$ or $\cos(\omega t)$.

We start with an example and then draw some general conclusions. Consider an object whose mass is 1 kgm stretches a spring 5 meters. The object is connected to a damper with damping constant $\gamma = 2$ and an external force equal to $F(t) = \cos t$ is also applied. The object is pushed up 4/5 meters above its equilibrium postion and then set into motion with a downward velocity of 0 meters/sec. Then determine the displacement $y(t)$ at any time $t \geq 0$.

The equation for total force now contains the additional external force: $my'' = mg - k(L + y) - \gamma y' + F(t)$

We simplify this as we did last time using the formula $mg = kL$ to obtain

$$my'' + \gamma y' + ky = F(t)$$

Note that the ODE is a before except the external driving force remains on the right hand side and hence is nonhomogeneous.

In our example which includes a damping device and external force, the IVP that we must solve is

$$y'' + 2y' + 2y = \cos t \qquad y(0) = -\frac{4}{5} \qquad y'(0) = 0$$

The characteristic polynomial is $r^2 + 2r + 2$. So the general solution to the associated homogeneous equation is $y_c = e^{-t}(c_1 \cos t + c_2 \sin t)$. We now seek a solution y_p to the given equation by complexifying it:

$$y'' + 2y' + 2y = e^{it}$$

We expect that a solution will be of the form $y_c = Ae^{it}$. We plug this in to the above to obtain

$$(-A + 2iA + 2A)e^{it} = e^{it}$$

Therefore $A = 1/(1 + 2i) = (1/5)(1 - 2i)$. We now take the real part of $\frac{e^{-t}}{5}(1 - 2i)(\cos t + i \sin t)$ to obtain a particular solution of the original equation:

$$y_p = \frac{1}{5}(\cos t + 2 \sin t)$$

So the general solution is $y_c + y_p$, where $y_c = e^{-t}(c_1 \cos t + c_2 \sin t)$ and the only thing that remains is to determine the c_1 and c_2:

$$y = \frac{1}{5}(\cos t + 2 \sin t) + e^{-t}(c_1 \cos t + c_2 \sin t) \qquad y' = \frac{1}{5}(-\sin t + 2 \cos t) - y_c' e^{-t}(-c_1 \sin t + c_2 \cos t)$$

Plugging $t = 0$ into these equation gives: $-4/5 = (1/5) + c_1 \qquad 0 = (1/5)(2) + 4/5 + c_2$ which gives $c_1 = -1$ and $c_2 = -6/5$.

We see two distinctive sorts of behaviors in y_c and y_p: Decaying oscillations in y_c and oscillations of "constant amplitude" in y_p. The technical names for these two types of behaviors are **transient** and **steady state**. This essentially says that after a short initial adjustment period the motion of the object behaves essentially in the same way as the external force.

We now consider the same spring-mass system except we assume that there is no damping and the external force is $F(t) = \cos(\sqrt{2}t)$

$$y'' + 2y = \cos 4t \qquad y(0) = -\frac{4}{5} \qquad y'(0) = 0$$

We again seek a general solution of the form $y = y_c + y_p$. The y_c is easily seen to be a linear combination of $\cos(\sqrt{2}t)$ and $\sin(\sqrt{2}t)$. We complexify in order to find a particular solution of the same ODE with right hand side equal to e^{4it}. The real part of y_d turns out to be a combination of $\sin 4t$ and $\cos 4t$. Therefore the final form of y will be a combination of 4 trig functions two having frequency 4 and two having frequency $\sqrt{2}$. This sort of combination exhibits the behavior that is known as **beats**.

We now consider the same spring-mass system except we assume that there is no damping and the external force is $F(t) = \cos(\sqrt{2}t)$

$$y'' + 2y = \cos \sqrt{2}t \qquad y(0) = -\frac{4}{5} \qquad y'(0) = 0$$

We again seek a general solution of the form $y = y_c + y_p$. The y_c is easily seen to be a linear combination of $\cos(\sqrt{2}t)$ and $\sin(\sqrt{2}t)$. We complexify in order to find a particular solution and we immediately see that constant next to the t in the exponential $e^{i\sqrt{2}t}$ is a root of the characteristic polynomial. This means that y_d must contain an extra factor of t and therefore its real and imaginary parts have this property as well. The extra factor of t appearing in y_p is is rather remarkable because it says that a bounded external force can cause unbounded oscillations in a spring-mass system. This phenomenon of dynamically storing energy in a spring-mass system is called **resonance**. It has both good and bad aspects to it from the view point of engineering and everyone must be cognizant of it.

From a mathematical point of view resonance occurs exactly when the frequency of the external force matches the natural frequency of the spring-mass system. However, the reality is that resonance has a spill over effect to an area called "near resonance" which is of interest to engineers.