

We continue with our analysis of a spring-mass system with the addition of a damping device. A damping device is a device that exerts a force on the object that is opposite in direction to its velocity and has magnitude that is proportional to the speed of the the object. A damping device can be a rather simple device and in fact it could be simply the air resistance exerted by the environment. (Of course, in our calculations last time we either assumed that the object was in an evacuated container or that air resistance was negligible.)

More precisely, the force due to the damping device is $-\gamma y'$, where γ is a constant called **damping constant** (or in the case of air resistance drag coefficient) and y' is the rate of change of displacement of the object, ie, its velocity. To incorporate this new ingredient into the ODE for y we place this new force among the forces comprising the total force on the object:

$$my'' = mg - k(L + y) - \gamma y'$$

We simplify this as we did last time using the formula $mg = kL$ to obtain

$$my'' + \gamma y' + ky = 0$$

Before we write down the solution for this ODE and examine its properties let us clarify what sort of behavior we expect. Last time we saw that once set into motion by releasing the object at an initial position α with an initial velocity β the object continues to oscillate periodically forever. It would seem that the presence of a weak damping device would not affect the motion of the object drastically but would tend to cause the oscillations to decrease. On the other hand, a strong damping device could conceivably eliminate all the oscillation the spring mass system is capable of. In fact, if one visualizes the suspension of a car in a very simplistic manner, then one is well aware of the fact that the vehicle safety inspection includes a test of this very property of the cars shock absorbers, namely, that they be strong enough to eliminate all oscillations after a car encounters a bump in the road.

Now let us write down characteristic polynomial of the damped spring-mass system: $mr^2 + \gamma r + k$. The roots of this are

$$\frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}$$

Note that by definition all the coefficients in the ODE are nonnegative.

If γ is small (and positive) then the stuff below the square root sign is negative and hence the roots are complex. In this case we see that the general solution is

$$y = e^{-\frac{\gamma}{2m}t}(c_1 \cos(\psi t) + c_2 \sin(\psi t)) \quad \text{where} \quad \psi = \frac{\sqrt{4mk - \gamma^2}}{2m}$$

This is no longer a periodic function. However, some features of periodicity do remain. For example the time between successive crossings of the equilibrium position remains constant and is in fact equal to π/ψ . In particular, there infinitely many crossings of the equilibrium position.

If γ is large then $\gamma > \sqrt{\gamma^2 - 4mk} > 0$ and consequently $0 > -\gamma + \sqrt{\gamma^2 - 4mk}$. This means that both roots r_1, r_2 of the characteristic polynomial are negative. We see that indeed the general solution approaches zero as $t \rightarrow \infty$ and that even with a very large initial velocity v_0 propelling the object on the spring towards its equilibrium position, the object will cross the equilibrium position at most once.

Finally we would like to determine the separating value of γ between the two behaviors outline above. That value of γ is called **critical damping**. Obviously is is located where the square root goes from being real to purely imaginary, ie, where it is zero:

$$\text{critical damping} = \sqrt{4mk}$$

Returning to thinking about a car's safety inspection, critical damping is the state in which the car passes but will not pass the inspection if the shock absorbers are even just a little bit weaker.

The general solution in this case is given by

$$y = e^{-\frac{\gamma}{2m}t}(c_1 + c_2 t)$$

Therefore, the general solution approaches zero as $t \rightarrow \infty$ and and that, even with a very large initial velocity propelling the object towards its equilibrium position, the object crosses its equilibrium position at most once.