We will model the mechanical vibrations found in a spring mass system. The textbook also deals with electric vibrations in a resistance capacitor circuit. However, from the differential equations point of view there is no difference between them and therefore we just consider the former.

We begin by recalling Hooke's law for an ideal spring: the magnitude of the restorative is force is proportional to the amount stretched. The proportionality constant is known as the **spring constant** and we will denote it by k. The direction of the force is obvious: It is opposite to the direction in which the spring is stretched and opposite to the direction in which the spring is compressed.

One important aspect of Hooke's law is that it fails rather dramatically when the spring is stretched "excessively" or when it is fatigued. Since this aspect is not related to the linear 2nd order ODE we just studied, we shall simply assume that Hooke's law is valid upto the breaking point. This no doubts leads to erroneous conclusions but this course is meant to provide mathematical background for rigorous engineering course and not as a substitute for them.

We now visualize a spring with one end attached to the ceiling and hanging downward. If an object of mass m kgrams is attached to the other end of the spring and it has a certain mass we assume that it stretches the spring L meters until the object is at equilibrium, ie, the magnitude of the upward force exerted on the object by the spring is equal to the magnitude of force gravity exerts on the object. In terms of the spring constant k we have

$$mg = kL \qquad k = \frac{L}{mg} \qquad \frac{k}{m} = \frac{g}{L} \tag{1}$$

The object is the set into motion by pulling it α meters below its equilibrium position and releasing it at time zero with a downward velocity of β meters/sec. We denote by $y(t)$ the displacement of the object from its equilibrium position at time t.

Let us try to write down an ODE and an IVP for the unknown function y. For this purpose note that we tacitly defined y to measure the positive distance to be below the equilibrium position. This means that gravity is a positive force and that whenever the spring is stretched it exerts a negative force. The source of the ODE that we write is Newton's second law which equates the mass times the acceleration of the object y" with the total force on the object. The total force for the time being consists of two forces: the force due to gravity and the resorative force of the spring. Eventually we will also consider a damping force, a periodic an external force, a discontinuous force and an "impulse" (collision).

So we equate my" with the sum of the force due to gravity and the restorative force:

$$my" = mg - k(L + y)$$

And in view of the above formula for the spring constant this is even simpler

$$my" = mg - kL - ky = -ky \qquad \text{or} \qquad my" + ky = 0$$

The IVP for this ODE is

$$my" + ky = 0 \qquad y(0) = \alpha \quad y'(0) = \beta$$

The general solution for this equation is:

$$y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \qquad \text{where} \omega_0 = \sqrt{\frac{k}{m}}$$

Also remember that $\frac{k}{m} = \frac{g}{L}$

We consider the following problem. An object whose mass is 0.5 kg stretches a spring .4 meters. The object is pushed up .2 meters above its equilibrium postion and then set into motion with a upward velocity of 1.0 meters/sec. Determine the displacement $y(t)$ of the object from its equilibrium position at any time $t > 0$. What is the maximum displacement from the equilibrium position? How much will the spring stretch? What initial velocitiy can be given if the spring is does not break upto 7/5 meter of stretching? How many times does the object pass through its equilibrium position?

Since $mg = kL$ and $g = 10, m = 1/2, L = 2/5$ we have $k = 25/2$, the IVP for the displacement $y = y(t)$ is

$$\frac{1}{2}y" + \frac{25}{2}y = 0 \qquad y(0) = -1/5 \quad y'(0) = -1.0\beta$$

We see that

$$y = c_1 \cos 5t + c_2 \sin 5t \qquad y' = 5(-c_1 \sin 5t + c_2 \cos 5t$$

Therefore $c_1 = -1/5$ and $c_2 = -1/5$.

To find the maximum displacement we use the following general formula if

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

then

$$y = R\cos(\omega_0 t - \delta) \qquad \text{where} \qquad R = \sqrt{c_1^2 + c_2^2} \qquad \delta = \arg(c_1 + ic_2)$$

In our case $\delta = \frac{5\pi}{4}$ and $R = \frac{\sqrt{2}}{5}$ The spring stretches this addition R meters beyond the .4 meters that it was stretched to the equilibrium position. This is approximately .7 meters, well below the breaking point.

Now let us see restriction must be placed on v_0 if the total stretch on the spring cannot be greater than 1.4 meters = 7/5meters. c_1 is determined as before but $c_2 = -v_0/5$. Therefore, we need to have

$$2/5 + \sqrt{(1/5)^2 + (v_0/5)^2} \le 7/5$$

Subtracting 2/5 and squaring both sides gives $1 + v_0^2 \le 5$ and this means that v_0 should be between -2 and 2meters/sec.