The Method of Undetermined Coefficients for the linear constant coefficient ODE equation $L[y] = g(t)$ deals with five possibilities for $g(t)$ listed below:

i. $g(t)$ is a polynomial of degree n, which may be any nonnegative integer.
ii. $g(t)$ is an exponential function $e^{\alpha t}$.
iii. $g(t)$ is either a sine or cosine function $\cos\beta t$, $\sin\beta t$.
iv. $g(t)$ is a product of **ii.** and **iii.** .
v. $g(t)$ is a product of **i.** and **iv.** .

If we are willing to consider the complexified ODE corresponding to $L[y] = g(t)$, then we saw last time that in each of the five cases the right hand side of the complexified ODE is a polynomial times an exponential and the solution method can be given the following exceedingly simple formulation: "plug in a generic polynomial of the **same degree** times the **same exponential**".

Unfortunately, we will have to deal with one (or two) exceptional situation(s). Suppose you were willing for the sake of simplicity to ignore the exceptions, relying on the fact that the the exceptions are just a minute fractions of all the ODE's that will be solved incorrectly? Unfortunately, some of the physical applications which will be discussed after the exam are modeled precisely by these exceptional ODE's. Their importance makes it impossible for us to get away with ignoring them!

The following example illustrates the exception referred to above: Consider the ODE: $y'' + y' - 2y = e^t$ We try to plug in $y_p = Ae^t$ to find that

$$\begin{aligned} -2(y_p &= Ae^t) \\ +1(y_p' &= Ae^t) \\ +1(y_p'' &= Ae^t) \end{aligned}$$

We see that we need to choose A so that $A(0)e^t = e^t$ which of course is impossible. This calculation in fact shows us that Ae^t is a solution to the associated homogeneous ODE (just in case we forgot that fact) and hence it is obvious that it is impossible for one and the same function to solve both the given **nonhomogeneous** and the **accociated homogeneous** ODE at the same time.

If one looks at the explanation given here, then one can see that by putting an extra factor of t into the guess for y_p one does produce the desired result, namely, an equation that can be solved for the constant A. So let us try plugging in $y_p = Ate^t$:

$$\begin{aligned} -2(y_p &= Ate^t) \\ +1(y_p' &= A(1+t)e^t) \\ 1(y_p'' &= A(1+(1+t))e^t) \end{aligned}$$

If this procedure is going to work it must be the case that all the t terms cancel each other. That in fact is the case here. The remaining terms are: $A(1+1+1)e^t = e^t$. Therefore we choose $A = \frac{1}{3}$ and indeed $y_p = \frac{1}{3}te^t$ is a solution to the given ODE.

Let us try several more examples in which this problem surfaces and see how in each case it can be overcome by just inserting an extra factor of t into our usual guess for a particular solution.

$$y'' + 169y = \sin(13t)$$

We complexify this ODE: $y'' + 169y = e^{13it}$ and check the roots of the characteristic polynomial $r^2 + 169$ to see that $13i$ is a root. Therefore we seek a particular $y_d = Ate^{13it}$. Plugging in gives:

$$\begin{aligned} 169(y_p &= Ate^{13it}) \\ +0(y_p' &= A(1+13it)e^{13it}) \\ 1(y_p'' &= A(13i+13i(1+13it))e^{13it}) \end{aligned}$$

Again looking for the t terms we find that they cancel out. The remaining terms are $A(13i+13i)e^{13it}$ which is equal to e^{13it} if $A = \frac{-i}{26}$. Therefore $y_d = \frac{-it}{26}(\cos(13t) + i\sin(13t))$ and taking the imaginary part gives us the solution to the original ODE $y_p = \frac{-t}{26}(\cos(13t)$

How about the following simple but important problem:

$$y'' + 2y' + 2y = e^{-t}\cos(t)$$

We complexify this ODE: $y'' + 2y' + 2y = e^{(-1+i)t}$ and check the roots of the characteristic polynomial $r^2 + 2r + 2$ to see that $-1 + i$ is a root. Therefore, we seek a particular $y_d = Ate(-1 + i)t$. Plugging this into the equation gives:

$$\begin{array}{rcl} 2(y_c & = & Ate^{(-1+i)t}) \\ 2(y_p' & = & A(1 + (-1 + i)t)e^{(-1+i)t}) \\ 1(y_p'' & = & A((-1 + i) + (-1 + i)(1 + (-1 + i)t))e^{(-1+i)t}) \end{array}$$

Adding up all terms involving t gives $A(2+2(-1+i)+(-1+i)^2)te^{(-1+1i)t}$ which is zero as expected. The remaining terms are $A(2+(-1+i)+(-1+i))e^{(-1+1i)t}$ which we set equal to $e^{(-1+i)t}$ to find that $A(2i)$ must be 1. Therefore,

$$y_d = \frac{-it}{2}e^{(-1+1i)t} = \frac{-it}{2}e^{-t}(\cos t + i\sin t)$$

Taking the real part of this expression gives: $y_p = \frac{1}{2}te^{-t}\sin t$

How about the following simple examples requires us to insert a factor of e^{0t} on the right hand side $y'' + y' = 3t + 4$ Normally we would guess that a linear polynomial could solve this ODE: $y_p = At + B$. However, this is not correct and we can see why by putting the right hand side in the usual form of a polynomial times an exponential and checking the roots of the characteristic polynomial:

$$y'' + y' = (t + 2)e^{0t}$$

and checking the roots of the characteristic polynomial $r^2 + r$ to see that that 0 and -1 are roots therefore the usual guess that y_p, a "polynomial of the same degree times the same exponential", will not work. We need to multiply by an extra factor of t: $y_p = (At + B)t = At^2 + Bt$ Plugging this into the ODE gives:

$$\begin{array}{rcl} 0(y_p & = & At^2 + Bt \\ +1(y_p' & = & 2At + B \\ +1(y_p'' & = & 2A) \end{array}$$

As expected there all the t^2 terms cancel. The t terms are $2At$ and we set it equal to $3t$, ie, $A = 3/2$ and the constant terms are $B + 2A$ and we set it equal to 4, ie, $B = 1$. We conclude that:$y_p = \frac{3}{2}t + 1$.

Finally let us look at one more simple example: $y'' + 2y' + y = e^{-t}$. Here the characteristic polynomial is $r^1 + 2r + 1$ which has a double root $r_1 = -1$. Therefore choosing e^{-t} for y_p will not work and also te^{-t} will also not work because it also solves the associated homogeneous ODE. The only thing that comes to mind in a situation like this is to try to multiply by an extra factor of t^2. Let us plug in to see if this works:

$$\begin{array}{rcl} 1(y_p & = & At^2e^{-t}) \\ +2(y_p' & = & A(2t - t^2)e^{-t}) \\ +1(y_p'' & = & A(2 - 2t - (2t - t^2))e^{-t}) \end{array}$$

To our amazement all the t^2 terms cancel and the same is true for all the t terms. A single term remains $2Ae^{-t}$ which is equal to e^{-t} if $A = 1/2$, ie, a particular solution is

$$y_p = \frac{1}{2}t^2e^{-t}$$

We are at the correct prescription for solving any nonhomogeneous equation linear constant coefficient ODE when the right hand side is a polynomial times an exponential is as follows:

Plug in a generic polynomial of the same degree times the same exponential, except when the constant in the exponential is a root of the characteristic polynomial. If it is a simple root multiply by an extra factor of t and if it is a double root then multiply by an extra factor of t^2.

So let us summarize everything we did for the past three days in the following **four** steps to solving the nonhomogeneous linear constant homogeneous ODE with the function $g(t)$ given by the five cases listed above. (We only discuss the case where g consists of single term because the case of g consisting of a sum of terms is handled by the applying the Superposition Principle.)

Step I: Complexify the ODE if needed. (i.e., If g contains a sine or cosine, then replace the given ODE with the complexified ODE, noting whether the original is the real or the imaginary part of the complexified ODE.)

Step II: Find the roots of the characteristic polynomial. (These are needed in any to find the general solution)

Step III: Take y_p or y_d to be a generic polynomial of the same degree as the one that appears as a factor in g times the same exponential as the one that appears as a factor in g.

Step IV: If the constant appearing in the exponential factor of g is a simple root of the characteristic polynomial then multiply y_p or y_d by t. And if it is a double root then multiply y_p or y_d by t^2.

Step V: Plug y_p or y_d into the ODE and solve for the constants. The number of constants to be determined is not altered in Step IV.) If you found y_d, then take the real or imaginary part for y_p.

One final comment. Since complex roots of characteristic polynomials occur in conjugate pairs, multiplication by t^2 will not be required when the original equation has trig functions on the right hand side.