We return to the "method of undetermined coefficients" which gives a procedure for finding a particular solution of nonhomogeneous second order linear constant coefficient ODE: equation $L[y] = g(t)$ for the five possibilities of $g(t)$ listed below

i. $g(t)$ is a polynomial of degree n, which may be any nonnegative integer.
ii. $g(t)$ is an exponential function $e^{\alpha t}$.

iii. $g(t)$ is either a sine or cosine function $\cos\beta t$, $\sin\beta t$.

iv. $g(t)$ is a product of **ii.** and **iii.** .

v. $g(t)$ is a product of **i.** and **iv.** .

Recall that yesterday we saw that cases **i.** and **ii.** are easily handled in a straight forward maner. In fact in the first case y_p can be taken to be polynomial of the same degree can be and in the second case a constant times the same exponential works. We also so that a product of a polynomial and exponential can be handle by taking y_p to be a generic object of the same form, ie, a product of a generic polynomial of the same degree times the same exponential. In fact, yesterday we really covered one simple case, NOT three separate cases. Indeed, a polynomial can always be viewed as a polynomial times the exponential e^{0t}. And, an exponential can always be viewed as a polynomial of degree zeor times that exponential. So, we were really talking only about an exponontial times a polynomial and our judicious guess at a particular solution was yesterday always a generic polynomial of the SAME degree times that SAME exponential.

We go on to the case where $g(t)$ is either $\cos\beta t$ or $\sin\beta t$. We begin by find a particular solution y_p of the ODE

$$y'' - 3y' + 4y = \sin(2t)$$

It is however much more efficient to first solve a somewhat different problem, called the complexified equation:

$$y'' - 3y' + 4y = e^{2it}$$

The connection between this and the original ODE can be see by recalling the definition $e^{2it} = \cos(2t) + i\sin(2t)$. Indeed the imaginary part of the right hand side of the complexified equation is the right hand side of the given ODE. So if we succeed in finding a solution y_d to the complexified ODE, then by the Superposition Principle

$$L[y_d] = L[\text{Re}y_d + i\text{Im}y_d] = L[\text{Re}y_d] + iL[\text{Im}y_d] = \cos(2t) + i\sin(2t)$$

Now, equating real and imaginary parts shows that

$$L[\text{Re}y_d] = \cos(2t) \qquad \text{and} \qquad L[\text{Im}y_d] = \sin(2t)$$

So we choose $y_d = Ae^{2it}$, plug it into the left hand side of the ODE and try to solve for A:

$$\begin{aligned} 4(y_d &= Ae^{2it}) \\ -3(y_p' &= 2iAe^{2it}) \\ 1(y_p' &= -4Ae^{2it}) \end{aligned}$$

Therefore, $(-6i)Ae^{2it} = e^{2it}$ and hence $A = \frac{1}{-6i}$ and $y_d = \frac{1}{-6i}e^{2it}$. The final step is to calculate the imaginary part of y_d and for this purpose we move the i out of denominator using the complex conjugate:

$$y_d = \frac{+i}{6}\left(\cos(2t) + i\sin(2t)\right)$$

Finally

$$y_p = \frac{1}{6}\cos(2t)$$

Note that we actually solved two ODE's while solving one. Specifically, we can easily write down the solution to

$$y'' - 3y' + 4y = \cos(2t)$$

by simply taking the real part of y_d.

The same technique can also be used to find particular solutions in the case where g is a product of an exponential function and trig function, ie, case **iv.** .

Consider the following example of this procedure:

$$y'' - 3y' + 4y = e^t \cos 2t$$

We realize that the complexification of this ODE is

$$y'' - 3y' + 4y = e^{(1+2i)t}$$

and the fact that the right hand side is simply an exponential makes it very easy to find a particular solution.

In fact we plug in $y_d = Ae^{(1+2i)t}$ into the complexified ODE:

$$\begin{aligned} 4(y_d &= Ae^{(1+2i)t}) \\ -3(y_p' &= (1+2i)Ae^{(1+2i)t}) \\ 1(y_p' &= (-3+4i)Ae^{1+2it}) \end{aligned}$$

Therefore, $(4 - 3(1+2i) - 3 + 4i)Ae^{(1+2i)t} = e^{(1+2i)t}$ and hence $A = \frac{1}{-2-2i}$ and $y_d = \frac{1}{-2-2i}e^{(1+2i)t}$. The final step is to calculate the imaginary part of y_d and for this purpose we move the i out of denominator using the complex conjugate:

$$y_d = \frac{-2+2i}{8}(\cos(2t) + i\sin(2t))$$

Finally, y_p is the real part of y_d

$$y_p = \frac{1}{4}(-\cos(2t) - \sin(2t))$$

Last time we saw that if the right hand side is a polynomial times an exponential, then y_p will turn out to be a polynomial of the same degree times the same exponential. This idea leads us to deal with case **v.** .

In fact let's try to find a particular solution to

$$y'' + y' - y = te^t \sin t$$

We first complexify:

$$y'' + y' - y = te^{1+i}$$

and choose

$$y_d = (At + B)e^{(1+i)t}$$

to plug back into the ODE:

$$\begin{aligned} -(y_d &= (At+B)e^{(1+i)t}) \\ 1(y_d' &= (A + (At+B)(1+i))e^{(1+i)t}) \\ 1(y_d'' &= (A(1+i) + (1+i)(A + (At+B)(1+i)))e^{(1+2i)t}) \end{aligned}$$

The above expression is complicated. So we approach it in two parts. We first look at all the terms involving t and set them equal to $te^{(1+i)t}$ and the sum of all remaining terms we set equal to zero. Adding up all terms involving t gives the equation

$$(-1 + (1+i) + (1+i)^2)A = 1$$

$$3iA = 1 \qquad \text{or} \qquad A = \frac{-i}{3}$$

Now, equating the remaining terms to zero gives

$$-B + A + B(1+i) + A(1+i) + (1+i)A + B(1+i)^2 = 0$$

Fortunately, we already know $A = -i/3$, and that simplifies the equation to:

$$-B - i/3 + B(1+i) - 2i(1+i)/3) + B(2i) = 0$$

$$-B - 3i/3 + 2/3 + B + iB + 2iB = 0$$

$$3iB = i - 2/3$$

$$B = 1/3 + 2i/9$$

Therefore

$$y_d = \left(-i\frac{t}{3} + \frac{1}{3} + i\frac{2}{9}\right) e^{(1+i)t} = \frac{-it}{3} e^t(\cos t + i \sin t) + \left(\frac{1}{3} + i\frac{2}{9}\right) e^t(\cos t + i \sin t)$$

And finally,

$$y_p = \text{Re} y_d = -te^{-t}(-t\cos t + 2\sin t)$$

$$y_p = \text{Re} y_d = e^t \left(-\frac{t}{3}\cos t + \frac{1}{3}\sin t + \frac{2}{9}\cos t\right)$$

In summary, we now have examples of how to solve all five possibilities for g. As we saw, if we complexify the ODE then there is really only one case to consider: g is a polynomial of degree n times an exponential e^{at}, with a a real or complex number.
So we saw that: Case **i.** is covered by this if we take $a = 0$.
Case **ii.** is covered by this if we take the polynomial to have degree 0
Case **iii.** is covered by this if we take the polynomial to have degree 0 and complexify the ODE
Case **iv.** is essentially the same as Case **iii.**
Case **v.** is covered by this if we complexify the ODE.

As a final example let us write down the form of the solution to plug in to solve the following nonhomogeneous ODE, but we will not solve to the unknown constants:

$$y'' - 8y' + 16y = (t^2 + 2t + 3)\sin 3t + (t+1)e^{4t}\cos 5t + (t^3 + t^2)e^{6t}$$

If one actually had to find a particular solution then it is advisable to break the above problem into 3 separate problems, taking g to be each of the terms appearing on the right hand side one at a time and then, by the SuperPosition Principle, adding the 3 pieces y_p found in this fashion together:

For $g = (t^2 + 2t + 3)\sin 3t$ one tries $y_d = (At^2 + Bt + C)e^{3it}$ and $y_p = \text{Im} y_d$.

For $g = (t+1)e^{4t}\cos 5t$ one tries $y_d = (At + B)e^{(4+5i)t}$ and $y_p = \text{Re} y_d$.

For $g = (t^3 + t^2)(e^{6t}$ one tries $y_p = ((At^3 + Bt^2 + Ct + D)e^{6t}$.